בס״ד

MASADA

סלעי ומצודתי

MY ROCK AND MY FORTRESS

2013
HIGHER SCHOOL CERTIFICATE
TRIAL EXAMINATION

Mathematics
Extension 2

General Instructions

- Reading Time - 5 minutes
- Working Time - 3 hours
- Write using a blue or black pen. Black pen is preferred
- Board approved calculators may be used
- A table of standard integrals is provided at the back of this paper
- Show all necessary working in Questions 11-16

Total marks (**100**)

Section I

Total marks (**10**)

- Attempt Questions 1-10
- Answer on the Multiple Choice answer sheet provided
- Allow about 15 minutes for this section

Section II

Total marks (**90**)

- Attempt questions 11 – 16
- Answer on the blank paper provided, unless otherwise instructed
- Start a new page for each question
- All necessary working should be shown for every question
- Allow about 2 hours 45 minutes for this section

Section I
10 marks
Attempt Questions 1-10
Allow about 15 minutes for this section

Use the multiple choice answer sheet for Questions 1 – 10.

1. Which of these expressions is equal to a square root of $8 + 6i$?

   (A) $3 - i$ (B) $5 - 3i$

   (C) $-3 - i$ (D) $-3 + i$

2. The equation of a curve is given by $x^2 + xy + y^2 = 9$. Which of the following expressions will provide the value of $\frac{dy}{dx}$ at any point on the curve?

   (A) $\frac{-2x - y}{2y}$ (B) $\frac{-2x - y}{x + 2y}$

   (C) $\frac{-2x + y}{2y}$ (D) $\frac{-2x + y}{x + 2y}$

3. The equation $x^3 + 2x^2 - 4x + 5 = 0$ has roots α, β and γ. Find the value of $a^2 + \beta^2 + \gamma^2$.

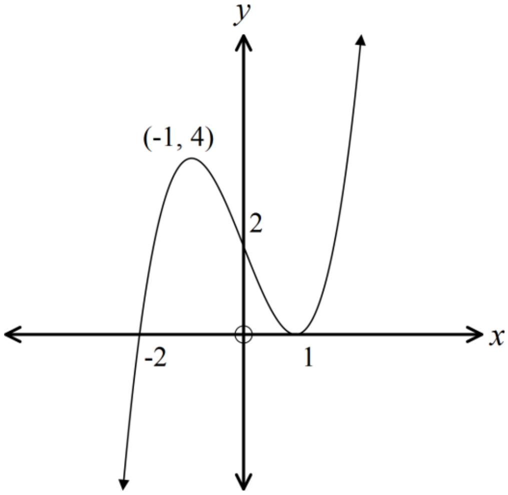
   (A) -12 (B) -4 (C) 4 (D) 12

4. The area bounded by the curves $y = x^2$ and $x = y^2$ is rotated about the x – axis. What is the volume of the solid of revolution formed?

   (A) $\frac{9\pi}{70}$ (B) $\frac{3\pi}{10}$

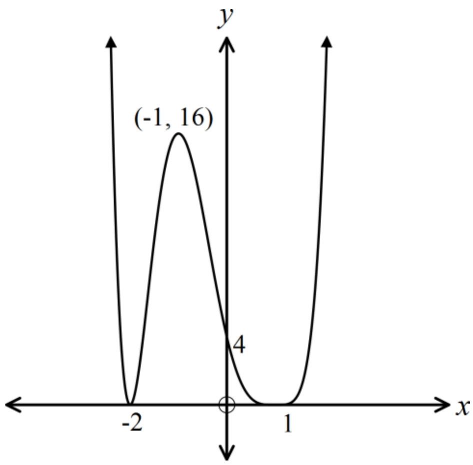
   (C) $\frac{7\pi}{10}$ (D) $\frac{3\pi}{2}$

5. The graph of the function $y = f(x)$ is drawn below:

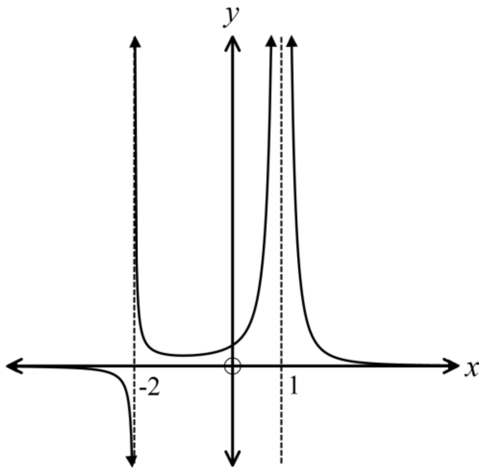


Which of the following graphs best represents the graph $y = \sqrt{f(x)}$?

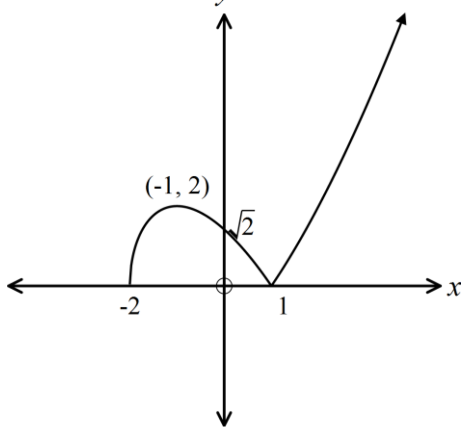
(A)



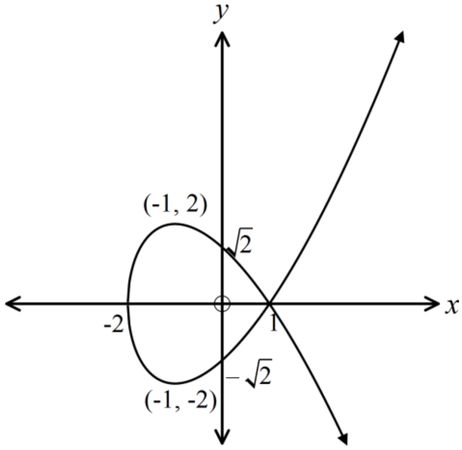
(B)



(C)



(D)



6. The equation of an hyperbola is given by $9x^2 - 4y^2 = 36$. Which of these represents the foci and the directrices of this hyperbola?

(A) $\left(\pm\sqrt{13},\, 0\right)$ and $x = \pm\dfrac{4\sqrt{13}}{13}$

(B) $\left(0,\, \pm\sqrt{13}\right)$ and $x = \pm\dfrac{4\sqrt{13}}{13}$

(C) $\left(\pm\sqrt{13},\, 0\right)$ and $y = \pm\dfrac{4\sqrt{13}}{13}$

(D) $\left(0,\, \pm\sqrt{13}\right)$ and $y = \pm\dfrac{4\sqrt{13}}{13}$

7. Which expression is equivalent to $\displaystyle\int \frac{x^3 - 1}{\left(x^4 - 4x\right)^{\frac{2}{3}}}\, dx$?

(A) $\dfrac{3}{4\sqrt[3]{x^4 - 4x}} + C$

(B) $\dfrac{3}{4\left(x^4 - 4x\right)} + C$

(C) $\dfrac{3\sqrt[3]{x^4 - 4x}}{4} + C$

(D) $\dfrac{3\left(x^4 - 4x\right)}{4} + C$

8. Josh has 6 balls in a box, 3 blue and 3 red. He chooses 2 balls at random and puts them in his pocket. One ball falls out of his pocket and he sees that it is blue. What is the probability that the other ball is also blue?

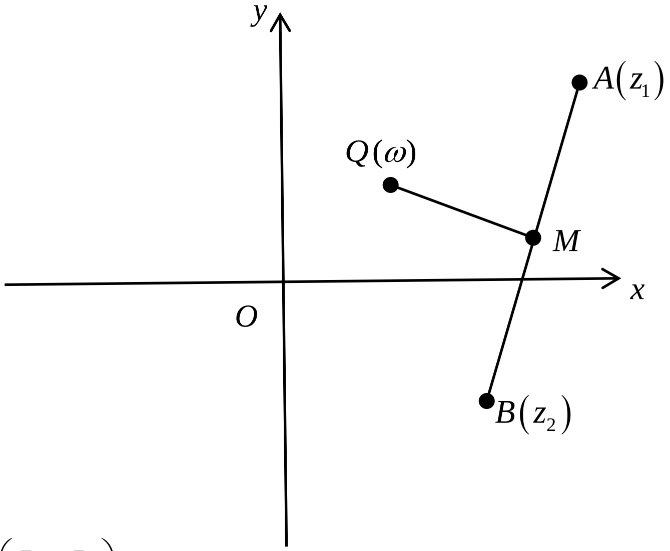
(A) $\dfrac{1}{5}$

(B) $\dfrac{1}{4}$

(C) $\dfrac{2}{5}$

(D) $\dfrac{1}{2}$

9. On the Argand diagram below, points A and B correspond to the complex numbers z_1 and z_2 respectively. M is the mid-point of the interval AB and QM is drawn perpendicular to AB. $QM = AM = BM$.
If Q corresponds to the complex number ω, what is the correct expression for ω ?



(A) $i\left(\frac{z_1 - z_2}{2}\right)$

(B) $i\left(\frac{z_1 + z_2}{2}\right)$

(C) $\frac{z_1 + z_2}{2} + i\left(\frac{z_1 + z_2}{2}\right)$

(D) $\frac{z_1 + z_2}{2} + i\left(\frac{z_1 - z_2}{2}\right)$

10. What is the solution to the inequation $2\sin 3x \geq 1$ if $0 \leq x \leq 2\pi$?

(A) $\frac{\pi}{6} \leq x \leq \frac{5\pi}{6}, \quad \frac{13\pi}{6} \leq x \leq \frac{17\pi}{6}, \quad \frac{25\pi}{6} \leq x \leq \frac{29\pi}{6}$

(B) $\frac{\pi}{6} \leq x \leq \frac{7\pi}{6}, \quad \frac{13\pi}{6} \leq x \leq \frac{20\pi}{6}, \quad \frac{25\pi}{6} \leq x \leq \frac{31\pi}{6}$

(C) $\frac{\pi}{18} \leq x \leq \frac{5\pi}{18}, \quad \frac{13\pi}{18} \leq x \leq \frac{17\pi}{18}, \quad \frac{25\pi}{18} \leq x \leq \frac{29\pi}{18}$

(D) $\frac{\pi}{18} \leq x \leq \frac{7\pi}{18}, \quad \frac{13\pi}{18} \leq x \leq \frac{20\pi}{18}, \quad \frac{25\pi}{18} \leq x \leq \frac{31\pi}{18}$

End of Section I

Section II

Total marks (90)
Attempt Questions 11-16
Allow about 2 hours 45 minutes for this section

Answer all questions, starting each question in a new writing booklet.

Question 11 **(15 marks)** Use a SEPARATE writing booklet **Marks**

(a) (i) If $\dfrac{x}{(x-2)^2(x-1)} = \dfrac{A}{x-2} + \dfrac{B}{(x-2)^2} + \dfrac{C}{x-1}$, find the values of A, B and C. **2**

(ii) Hence find $\displaystyle\int \frac{x}{(x-2)^2(x-1)}\,dx.$ **2**

(b) (i) Derive the reduction formula **1**

$$\int x^n e^x\,dx = x^n e^x - n\int x^{n-1} e^x\,dx.$$

(ii) Hence or otherwise, find $\displaystyle\int_0^2 x^3 e^x\,dx$ **2**

(c) Find

(i) $\displaystyle\int \frac{5x-3}{x^2+6x-14}\,dx$ **3**

(ii) $\displaystyle\int \frac{dx}{\left(25+x^2\right)^{\frac{3}{2}}}$. using the trigonometric substitution $x = 5\tan\theta$. **3**

(iii) $\displaystyle\int \frac{dx}{\sqrt{4+2x-x^2}}$. **2**

End of Question 11

Question 12 (15 marks) Use a SEPARATE writing booklet **Marks**

(a) If $z = \dfrac{1+\sqrt{3}\,i}{\sqrt{3}+i}$, find :

(i) $|z|$ **2**

(ii) Show that $Arg\,(z) = \dfrac{\pi}{6}$ **1**

(iii) z^4 **1**

(iv) the five fifth roots of z. **2**

(b) Sketch the region in the Argand plane consisting of those points z for which: **3**

$$|\arg(z+1)| < \frac{\pi}{6},\quad z + \bar{z} \le 6 \text{ and } |z+1| > 2.$$

There is no need to find points of intersection.

(c)

(i) Use De Moivre's theorem to express $\cos 4\theta$ in terms of $\cos\theta$. **2**

(ii) By obtaining a similar expression for $\sin 4\theta$, show that **2**

$\cot 4\theta = \dfrac{x^4 - 6x^2 + 1}{4x^3 - 4x}$ where $x = \cot\theta$.

(iii) By considering the roots of $\cot 4\theta = 0$ prove that **2**

$$\cot\frac{\pi}{8}\,.\,\cot\frac{3\pi}{8}\,.\,\cot\frac{5\pi}{8}\,.\,\cot\frac{7\pi}{8} = 1.$$

End of Question 12

Question 13 (15 marks) Use a SEPARATE writing booklet **Marks**

(a)

(i) Show that if $x = a$ is a double root of the polynomial $P(x) = 0$, then $P'(a) = P(a) = 0$. **2**

(ii) Find the roots of the equation $f(x) = x^4 - 2x^3 + x^2 + 12x + 8$, given that it has a double root. **3**

(b) Given that one root of the equation $x^4 - 5x^3 + 5x^2 + 25x - 26 = 0$ is $3 + 2i$, solve the equation. **3**

(c) The equation $2x^3 - x^2 + 3x - 1 = 0$ has roots α, β, and γ. Find the equation which has roots:

(i) $2\alpha, 2\beta$ and 2γ. **2**

(ii) α^2, β^2 and γ^2. **2**

(d) The region bounded by $y = \sin x$ and the x - axis between $x = 0$ and $x = \pi$ is rotated about the y - axis. **3**

Find the volume of the resulting solid, using the method of cylindrical shells.

End of Question 13

Question 14 (15 marks) Use a SEPARATE writing booklet **Marks**

(a) (i) Show that the equation of the locus of a point $P(x, y)$ that moves such that the sum of its distances from $A(0, 3)$ and $B(0, -3)$ is 10 units is given by $\frac{x^2}{16} + \frac{y^2}{25} = 1$. **4**

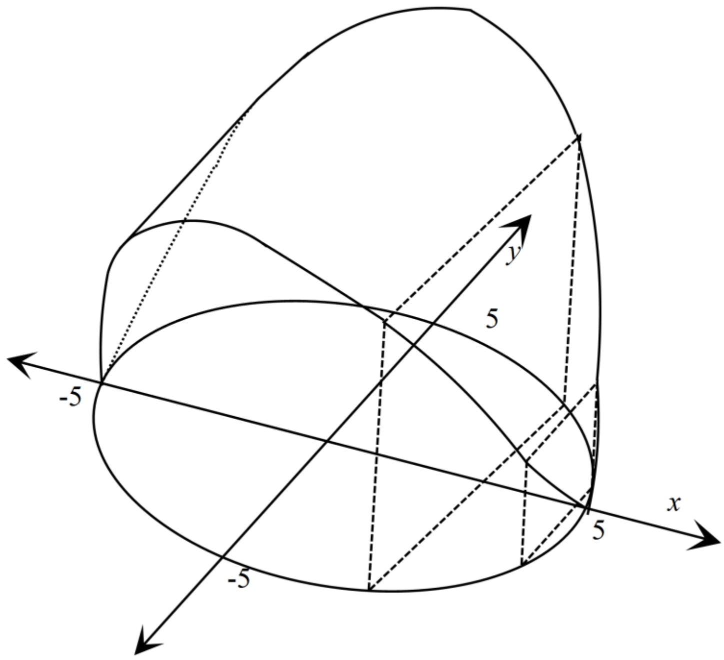
(ii) Find the equations of the tangents to the ellipse when $y = 4$. **3**

(b) A point $P\left(ct, \frac{c}{t}\right)$ lies on the rectangular hyperbola $xy = c^2$.

(i) Show that the equation of the tangent at the point $P\left(ct, \frac{c}{t}\right)$ on the rectangular hyperbola is given by $x + t^2 y = 2ct$. **2**

(ii) Prove that the area bounded by the tangent and the asymptotes of the hyperbola is a constant. **2**

(c) Let S be the solid having for its base the region bounded by the circle $x^2 + y^2 = 25$. **4**



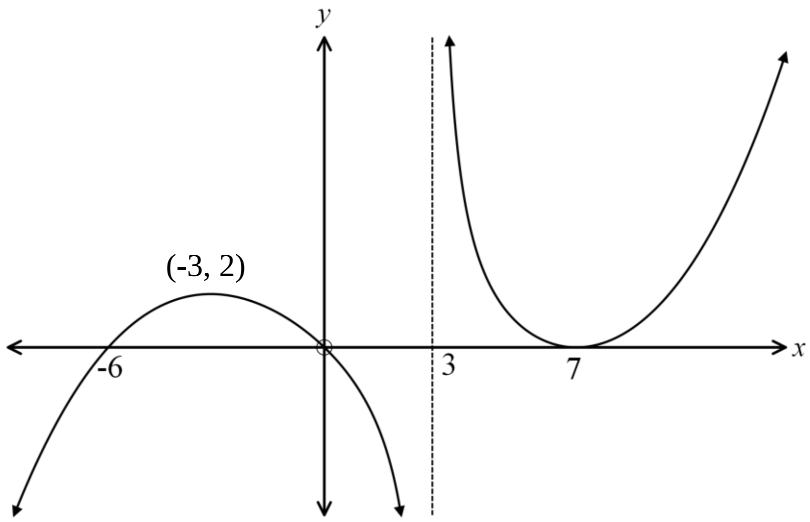
Every vertical plane section of the solid perpendicular to the x- axis is a square.

Find the volume of the solid S.

End of Question 14

Question 15 (15 marks) Use a SEPARATE writing booklet **Marks**

(a) The graph of is $y = f(x)$ shown below:



Draw separate half page sketches of the following (indicate important features).

(i) $y = \dfrac{1}{f(x)}$. **2**

(ii) $y = | f(x) |$. **2**

(iii) $y = e^{f(x)}$. **3**

(b) (i) Use the principle of Mathematical Induction to prove that, for all positive integers i: **3**

$$\sum_{i=1}^{n} i^2 = \frac{n^3}{3} + \frac{n^2}{2} + \frac{n}{6}.$$

(ii) Hence evaluate $\lim\limits_{x \to \infty} \dfrac{1^2 + 2^2 + 3^2 + \ldots\ldots + n^2}{n^3}$ **1**

(c) A particle of mass one kg is moving in a straight line. It is initially at the origin and is travelling with velocity $\sqrt{3}\,ms^{-1}$. **4**

The particle is moving against a resistance $v + v^3$, where v is the velocity. Show that the displacement x of the particle from the origin is given by $x = \tan^{-1}\left(\dfrac{\sqrt{3} - v}{1 + v\sqrt{3}}\right)$.

End of Question 15

Question 16 (15 marks) Use a SEPARATE writing booklet **Marks**

(a) A body of mass 50 kg falls from a height at which gravitational acceleration is g. Assuming that air resistance is proportional to the speed v with a constant of proportion being $\frac{1}{10}$, find :

(i) The velocity after time t. **3**

(ii) The terminal velocity. **1**

(iii) The distance the object has fallen after time t. **2**

(b) (i) Show that $y = x - 1$ is a tangent to the curve $y = \log_e x$ at the point where $x = 1$. **1**

(ii) Hence, or otherwise, show that $\log_e x \le x - 1$ for $x > 0$. **2**

(iii) Given n positive numbers $a_1, a_2, a_3, \ldots\ldots, a_n$ such that **2**

$a_1 + a_2 + a_3 + \ldots\ldots + a_n = 1$, prove that $\sum_{k=1}^{n} \log_e(na_k) \le 0$.

(iv) Hence show that $a_1 a_2 a_3 \ldots\ldots a_n \le \frac{1}{n^n}$. **2**

(v) Given n positive numbers $x_1, x_2, x_3, \ldots\ldots, x_n$, **2**

prove that $x_1 x_2 x_3 \ldots\ldots x_n \le \left[\frac{1}{n}\left(x_1 + x_2 + x_3 + \ldots\ldots + x_n\right)\right]^n$

End of Examination

STANDARD INTEGRALS

$$\int x^n \, dx = \frac{1}{n+1} x^{n+1}, \quad n \neq -1; \quad x \neq 0, \text{ if } n < 0$$

$$\int \frac{1}{x} dx = \ln x, \quad x > 0$$

$$\int e^{ax} \, dx = \frac{1}{a} e^{ax}, \quad a \neq 0$$

$$\int \cos ax \, dx = \frac{1}{a} \sin ax, \quad a \neq 0$$

$$\int \sin ax \, dx = -\frac{1}{a} \cos ax, \quad a \neq 0$$

$$\int \sec^2 ax \, dx = \frac{1}{a} \tan ax, \quad a \neq 0$$

$$\int \sec ax \, \tan ax \, dx = \frac{1}{a} \sec ax, \quad a \neq 0$$

$$\int \frac{1}{a^2 + x^2} dx = \frac{1}{a} \tan^{-1} \frac{x}{a}, \quad a \neq 0$$

$$\int \frac{1}{\sqrt{a^2 - x^2}} dx = \sin^{-1} \frac{x}{a}, \quad a > 0, \quad -a < x < a$$

$$\int \frac{1}{\sqrt{x^2 - a^2}} dx = \ln\left(x + \sqrt{x^2 - a^2} \right), \quad x > a > 0$$

$$\int \frac{1}{\sqrt{x^2 + a^2}} dx = \ln\left(x + \sqrt{x^2 + a^2} \right)$$

NOTE : $\ln x = \log_e x, \quad x > 0$