**Question One:** (15 Marks) *Start a new sheet of paper.*

a) Find $\int \frac{x}{\sqrt{2-x^2}}\,dx$ using the substitution $x=\sqrt{2}\sin\theta$. [2]

b) Show that $\sin(A+B)+\sin(A-B)=2\sin A\cos B$, and hence find

$\int \sin 5x \cos 3x\,dx$. [3]

c) Use Integration by Parts to show that $\int_0^1 \tan^{-1} x\,dx = \frac{\pi}{4}-\frac{1}{2}\ln 2$. [3]

d) Given that $J_n = \int_0^{\frac{\pi}{2}} \cos^n x\,dx$ :

i) Prove that $J_n = \frac{(n-1)}{n} J_{n-2}$, where $n$ is an integer and $n \geq 2$. [4]

ii) Hence evaluate $\int_0^{\frac{\pi}{2}} \cos^6 x\,dx$. [3]

**Question Two:** (15 Marks) *Start a new sheet of paper.*

a) Given that $z=2+i$ and $\omega=2-3i$, find, in the form $a+ib$

i) $(\bar{z})^2$ [1]

ii) $\left(\frac{z}{\omega}\right)$ [1]

b) On the Argand diagram, shade the region where the inequalities

$0<|z|<1$ and $\frac{\pi}{4}<\arg(z)<\frac{3\pi}{4}$ both hold. [3]

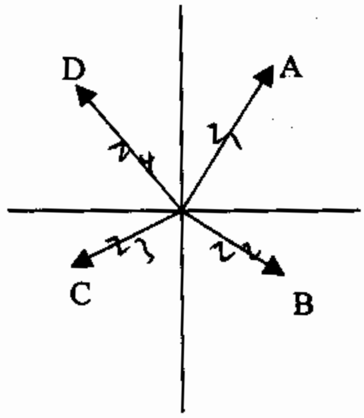
c) Find the complex square roots of $7+6i\sqrt{2}$, giving your answer in the form $a+ib$, where $a$ and $b$ are real. [3]

*(Question 2 continued over)*

d) Given the two complex numbers $z_1 = r_1 cis\theta$ and $z_2 = r_2 cis\phi$,

i) Show that, if $z_1$ and $z_2$ are parallel, $z_1 = kz_2$, for $k$ real. [1]

ABCD is a quadrilateral with vertices A, B, C and D represented by the complex numbers (vectors) $z_1$, $z_2$, $z_3$ and $z_4$, as shown in the sketch opposite.




ii) Give two possible vectors (in terms of $z_1, z_2$) for side AB. [1]

iii) If ABCD is a parallelogram, show that $z_1 - z_2 - z_3 + z_4 = 0$. [3]

e) Explain the fallacy in the following argument: [2]

$-1 = \sqrt{-1} \times \sqrt{-1} = \sqrt{(-1)(-1)} = \sqrt{1} = 1$. Hence $1 = -1$.

**Question Three:** (15 Marks) *Start a new sheet of paper.*

a) $F(x)$ is defined by the equation $f(x) = x^2\left(x - \frac{3}{2}\right)$, on the domain $-2 \le x \le 2$.

Note: each sketch below should take about one third of a page.

i) Draw a neat sketch of $F(x)$, labelling all intersections with coordinate axes and turning points. [2]

ii) Sketch $y = \dfrac{1}{F(x)}$ [2]

iii) Sketch $y = \sqrt{F(x)}$ [2]

iv) Sketch $y = \ln(F(|x|))$ [2]

b) The Hyperbola $\mathcal{H}$ has the equation $\frac{x^2}{25} - \frac{y^2}{9} = 1$.

i) Find the eccentricity of $\mathcal{H}$. [1]

ii) Find the coordinates of the foci of $\mathcal{H}$. [1]

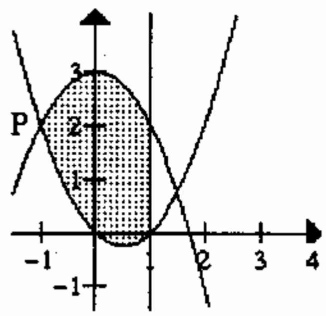
iii) Draw a neat one third of a page size sketch of $\mathcal{H}$. [2]

iv) The line $x = 6$ cuts $\mathcal{H}$ at A and B. Find the coordinates of A and B if A is in the first quadrant. [1]

v) Derive the equation of the tangent to $\mathcal{H}$ at A. [2]

**Question Four:** (15 Marks) *Start a new sheet of paper.*

a) The shaded region bounded by $y = 3 - x^2$, $y = x^2 - x$ and $x = 1$ is rotated about the line $x = 1$. The point P is the intersection of $y = 3 - x^2$ and $y = x^2 - x$ in the second quadrant.




i) Find the $x$ coordinate of P. [1]

ii) Use the method of cylindrical shells to express the volume of the resulting solid of resolution as an integral. [3]

iii) Evaluate the integral in part (ii) above. [2]

*(Question 4 continued over)*

b) Find real numbers A, B and C such that
$$\frac{x}{(x-1)^2(x-2)}=\frac{A}{(x-1)}+\frac{B}{(x-1)^2}+\frac{C}{(x-2)}.$$ [2]

Hence show that $\displaystyle\int_0^{\frac{1}{2}}\frac{x}{(x-1)^2(x-2)}dx=2\ln\left(\frac{3}{2}\right)-1$. [2]

c) Find all $x$ such that $\cos 2x=\sin 3x$, if $0\le x\le\frac{\pi}{2}$. [2]

d) Solve for $x$: $\tan^{-1}(3x)-\tan^{-1}(2x)=\tan^{-1}\left(\frac{1}{5}\right)$ [3]

**<u>Question Five:</u>** (15 Marks) *Start a new sheet of paper.*

a) For the polynomial equation $x^3+4x^2+2x-3=0$ with roots $\alpha,\beta$ and $\gamma$, find:

i) The value of $\alpha^2+\beta^2+\gamma^2$ [1]

ii) The equation whose roots are $(1-\alpha),(1-\beta),(1-\gamma)$. [2]

iii) The equation whose roots are $\frac{1}{\alpha},\frac{1}{\beta},\frac{1}{\gamma}$. [3]

b) Determine all the roots of $8x^4-25x^3+27x^2-11x+1=0$ given that it has a root of multiplicity 3. [4]

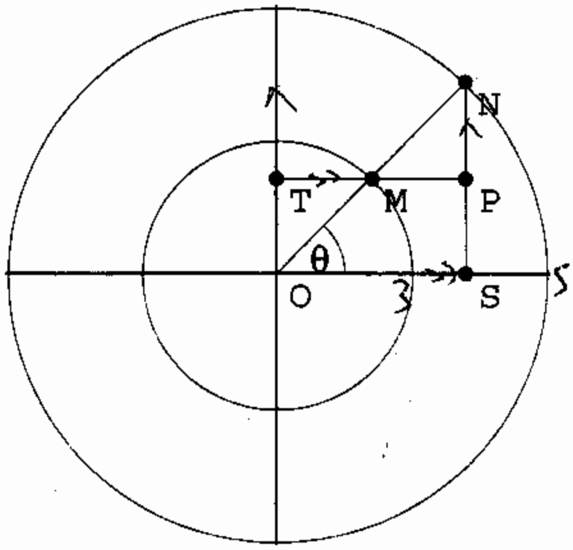
c) The equation $x^4+4x^3+5x^2+2x-20=0$ has roots $\alpha,\beta,\gamma$ and $\delta$ over the complex field.

i) Show that the equation whose roots are $\alpha+1,\beta+1,\gamma+1$ and $\delta+1$ is given by $x^4-x^2-20=0$. [2]

ii) Hence solve the equation $x^4+4x^3+5x^2+2x-20=0$. [3]

**Question Six:** (15 Marks) *Start a new sheet of paper.*

a)




The circles above have centres at O and radii of 5 units and 3 units respectively.

A ray from O making an angle $\theta$ with the positive $x$-axis, cuts the circles at the points M and N as shown.

NS is drawn parallel to the $y$-axis and MT parallel to the $x$-axis.

NS and MT intersect at P.

i) Show that the parametric equations of the locus of P in terms of $\theta$ are given by $x = 5\cos\theta$ and $y = 3\sin\theta$. [2]

ii) By eliminating $\theta$, find the Cartesian equation of this locus. [1]

iii) Find the equation of the normal (in general form) at the point P when $\theta = \frac{\pi}{3}$. [2]

b) The functions $S(x)$ and $C(x)$ are defined by the formulae

$S(x) = \frac{1}{2}\left(e^{x} - e^{-x}\right)$ and $C(x) = \frac{1}{2}\left(e^{x} + e^{-x}\right)$.

i) Verify that $S'(x) = C(x)$. [1]

ii) Show that $S(x)$ is an increasing function for all real $x$. [1]

iii) Prove $[C(x)]^2 = 1 + [S(x)]^2$ [2]

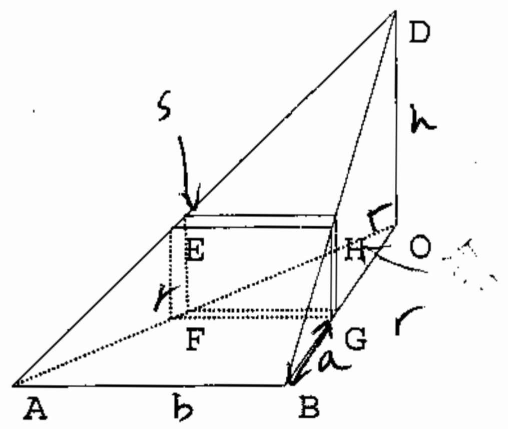
iv) $S(x)$ has an inverse function, $S^{-1}(x)$, for all real values of $x$. Briefly justify this statement. [1]

v) Let $y = S^{-1}(x)$. Prove that $\dfrac{dy}{dx} = \dfrac{1}{\sqrt{1+x^2}}$. [2]

vi) Hence, or otherwise, show that $S^{-1}(x) = ln\{x + \sqrt{1+x^2}\}$. [3]

**Question Seven:** (15 Marks) *Start a new sheet of paper.*

a) Let OAB be an isosceles triangle, OA = OB = r, AB = b.




Let OABD be a triangular pyramid with height OD = h and OD perpendicular to the plane of OAB as in the diagram above.

Consider a slice S of the pyramid of width $\delta a$ as shown at EFGH in the diagram. The slice S is perpendicular to the plane of OAB at FG with FG || AB and BG = a. Note that GH || OD.

i) Show that the volume of S is $\left(\dfrac{r-a}{r}\right)b\left(\dfrac{ah}{r}\right)\delta a$ when $\delta a$ is small.

(You may assume the slice is approximately a rectangular prism of base EFGH and height $\delta a$). [3]

ii) Hence show that the pyramid DOAB has a volume of $\dfrac{1}{6}hbr$. [2]

iii) Suppose now that $\angle AOB = \frac{2\pi}{n}$ and that $n$ identical pyramids DOAB are arranged about O as the centre with common vertical axis OD to from a solid C. Show that the volume $V_n$ of C is given by $V_n = \frac{1}{3}r^2hn\sin\frac{\pi}{n}$. [2]

iv) Note that when $n$ is large, C approximates a right circular cone. Hence, find $\lim_{n\to\infty} V_n$ and verify a right circular cone of radius r and height h has a volume of $\frac{1}{3}\pi r^2 h$. [2]

b) On the hyperbola $xy = c^2$, three points P, Q and R are on the same branch, with parameters $p, q$ and $r$ respectively. The tangents at P and Q intersect at U. If O, U and R are collinear, find the relationship between $p, q$ and $r$. [6]

**Question Eight:** (15 Marks) *Start a new sheet of paper.*

a)

i) Use the substitution $x = \frac{2}{3}\sin\theta$ to prove that $\int_0^{\frac{2}{3}} \sqrt{4-9x^2}\,dx = \frac{\pi}{3}$. [3]

ii) Hence, or otherwise, find the area enclosed by the ellipse

$9x^2 + y^2 = 4$. [1]

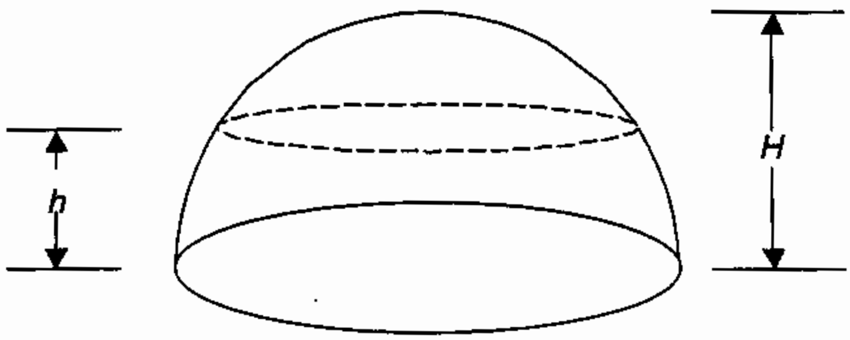
b)

i) Use an appropriate substitution to verify that $\int_0^a \sqrt{a^2 - x^2}\,dx = \frac{\pi a^2}{4}$. [2]

ii) Deduce that the area enclosed by the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ is given

by $\pi ab$. [2]

c) The diagram below shows a mound of height $H$. At height $h$ above the horizontal base, the horizontal cross-section of the mound is elliptical in shape, with equation $\frac{x^2}{a^2}+\frac{y^2}{b^2}=\lambda^2$, where $\lambda = 1-\frac{h^2}{H^2}$, and $x$, $y$ are appropriate coordinates in the plane of the cross-section.




Show that the volume of the mound is $\frac{8\pi abH}{15}$. [3]

d) The quadratic equation $x^2-(2\cos\theta)x+1=0$ has roots $\alpha$ and $\beta$.

i) Find expressions for $\alpha$ and $\beta$. [1]

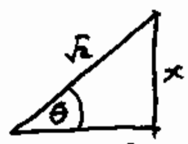
ii) Show that $\alpha^{10}+\beta^{10}=2\cos(10\theta)$. [3]

## UESTION ONE:

a) $x = \sqrt{2}\sin\theta$ so $dx = \sqrt{2}\cos\theta . d\theta$

$$\therefore \int \frac{x}{\sqrt{2-x^2}}\,dx$$
$$= \int \frac{\sqrt{2}\sin\theta . \sqrt{2}\cos\theta . d\theta}{\sqrt{2-(\sqrt{2}\sin\theta)^2}}$$
$$= \int \frac{2\sin\theta\cos\theta\,d\theta}{\sqrt{2(1-\sin^2\theta)}}$$
$$= \int \frac{\sqrt{2}\sin\theta\,\cancel{\cos\theta}\,d\theta}{\cancel{\cos\theta}}$$
$$= \int \sqrt{2}\sin\theta\,d\theta$$
$$= -\sqrt{2}\cos\theta + c$$
$$= -\sqrt{2} \cdot \frac{\sqrt{2-x^2}}{\sqrt{2}} + c$$
$$= -\sqrt{2-x^2} + c$$

also




with $\sin\theta = \frac{x}{\sqrt{2}}$

$x^2 + y^2 = 2$

$y^2 = 2 - x^2$

$y = \sqrt{2-x^2}$

$\therefore \cos\theta = \frac{\sqrt{2-x^2}}{\sqrt{2}}$

Marking: ① converting from $x$ to $\theta$, + resolving for $\cos\theta$. ① answer

Comments:
- Some got $dx = \sqrt{2}\cos\theta\,d\theta$, but then left it out of the substitution when changing to $\theta$!!
- very common error was not expressing the indefinite integral back in terms of $x$; $-\sqrt{2}\cos\theta + c$ got 1 mark.
- $-\sqrt{2}\cos\left(\sin^{-1}\left(\frac{x}{\sqrt{2}}\right)\right) + c$ was also not sufficient for both marks (not simplest form).

b) $\sin(A+B) = \sin A\cos B + \sin B\cos A$ — ①

$\sin(A-B) = \sin A\cos B - \sin B\cos A$ — ②

$\sin(A+B) + \sin(A-B) = 2\sin A\cos B$ ①+②

$$\therefore \int \sin 5x\cos 3x\,dx$$
$$= \int \tfrac{1}{2}\left(\sin(5x+3x) + \sin(5x-3x)\right)dx$$
$$= \int \tfrac{1}{2}\sin 8x + \tfrac{1}{2}\sin 2x\,dx$$
$$= -\tfrac{1}{16}\cos 8x - \tfrac{1}{4}\cos 2x + c$$

Marking: ① showing relationship. ① correct use of formula. ① Answer

Comments: · well done

)
$$\int_0^1 \tan^{-1}x\,dx$$
$$= \int_0^1 \frac{d(x)}{dx} \cdot \tan^{-1}x\,dx$$
$$= \left[x\tan^{-1}x\right]_0^1 - \int_0^1 x \cdot \frac{1}{1+x^2}\,dx$$
$$= \left(\tfrac{\pi}{4} - 0\right) - \tfrac{1}{2}\int_0^1 \frac{2x}{1+x^2}\,dx$$
$$= \tfrac{\pi}{4} - \tfrac{1}{2}\left[\ln(1+x^2)\right]_0^1$$
$$= \tfrac{\pi}{4} - \tfrac{1}{2}(\ln 2 - \ln 1)$$
$$= \tfrac{\pi}{4} - \tfrac{1}{2}\ln 2$$

Marking: ① correct integration by parts. ① correct log integration. ① correct algebra

Comments: · well done

SOLUTIONS: Yr 12 TRIAL HSC EXTN II: 2003 — MARKING — COMMENTS — P

1) i)
$$J_n = \int_0^{\frac{\pi}{2}} \cos^n x\,dx$$
$$= \int_0^{\frac{\pi}{2}} \cos x \cos^{n-1} x\,dx$$
$$= \left[\sin x \cos^{n-1} x\right]_0^{\frac{\pi}{2}} - \int_0^{\frac{\pi}{2}} (n-1)\cos^{n-2} x \cdot \sin x \cdot \sin x\,dx$$
$$= (0-0) + (n-1)\int_0^{\frac{\pi}{2}} \cos^{n-2} x \sin^2 x\,dx$$
$$= (n-1)\int_0^{\frac{\pi}{2}} (1-\cos^2 x)\cos^{n-2} x\,dx$$
$$= (n-1)\int_0^{\frac{\pi}{2}} \cos^{n-2} x\,dx - (n-1)\int_0^{\frac{\pi}{2}} \cos^n x\,dx$$
$$J_n = (n-1)\,J_{n-2} - (n-1)\,J_n$$
$$J_n + (n-1)J_n = (n-1)J_{n-2}$$
$$n\,J_n = (n-1)\,J_{n-2}$$
$$\therefore J_n = \frac{(n-1)}{n} J_{n-2}$$

Marking: (1) correct method for splitting cos; (1) resolving to integral; (1) resolving to $J_{n-2}$; (1) correct algebra to solution

Comments: • many tried the approach $\int \frac{d(x)}{dx} \cdot \cos^n x\,dx$ and got lost. These need to be known.

ii)
$$\therefore \int_0^{\frac{\pi}{2}} \cos^6 x\,dx = J_6$$
$$J_6 = \frac{5}{6} \cdot J_4$$
$$= \frac{5}{6} \cdot \frac{3}{4} J_2$$
$$= \frac{5}{6} \cdot \frac{3}{4} \cdot \frac{1}{2} J_0$$
$$= \frac{15}{48} \cdot \int_0^{\frac{\pi}{2}} dx$$
$$= \frac{15}{48} \cdot \left[x\right]_0^{\frac{\pi}{2}}$$
$$= \frac{15\pi}{96}$$

Marking: (1) correct use of formula; (1) evaluates $J_0$; (1) answer

Comments: • well done.

## QUESTION TWO:

a) i)
$$(\bar{z})^2$$
$$= (2-i)^2$$
$$= 4 - 4i - 1$$
$$= 3 - 4i$$

Marking: (1) answer

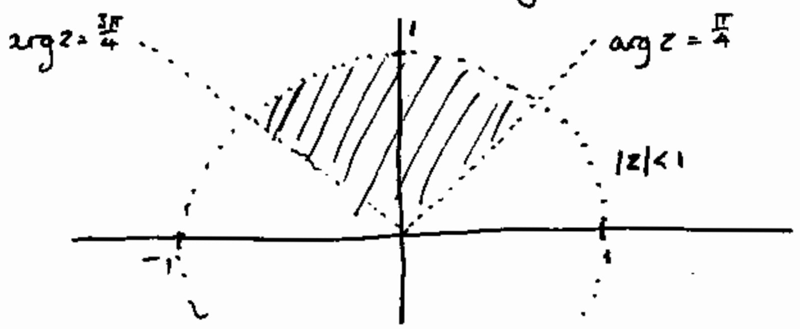
Comments: • some simple mistakes made

ii)
$$\left(\frac{\bar{z}}{\bar{w}}\right)$$
$$= \frac{(2+i)}{(2-3i)} \cdot \frac{(2+3i)}{(2+3i)}$$
$$= \frac{4+6i+2i-3}{4+9}$$
$$= \frac{1+8i}{13}$$

Marking: (1) answer

Comments: · generally good

iii) $0 < |z| < 1$ $\quad \frac{\pi}{4} < \arg z < \frac{3\pi}{4}$




Marking: (1) boundaries; (1) correct |z|; (1) correct arg limits.

Comments: · generally good.

c) $(a+ib) = \sqrt{7+6i\sqrt{2}}$ a, b real

$\therefore (a+ib)^2 = 7+6i\sqrt{2}$

$a^2-b^2+2abi = 7+6\sqrt{2}i$

equating real and imaginary parts.

$a^2-b^2=7$ —① $\quad 2ab = 6\sqrt{2}$

$\therefore a = \frac{6\sqrt{2}}{2b} = \frac{3\sqrt{2}}{b}$ —②

① setup a,b relationship

. mostly well done. Some students tried to use formulas for finding square roots of complex numbers (not very successfully)

substituting ② in ①:

$\left(\frac{3\sqrt{2}}{b}\right)^2 - b^2 = 7$

$\frac{18}{b^2} - b^2 = 7$

$18 - b^4 = 7b^2$

ie $b^4+7b^2-18=0$

$\therefore (b^2+9)(b^2-2)=0$

$b^2 = 2, -9$

① resolve for correct b values

reject $b^2=-9$ as b is real.

$\therefore b = \pm\sqrt{2}$ in ②:

$b=\sqrt{2}$, $a = \frac{3\sqrt{2}}{\sqrt{2}} = 3$

$b=-\sqrt{2}$, $a = \frac{3\sqrt{2}}{-\sqrt{2}} = -3$

① correct roots.

$\therefore$ roots are $3+\sqrt{2}i$, $-3-\sqrt{2}i$

---

l) i) for $z_1 \parallel z_2$, $\theta = \phi$, $\therefore z_2 = r_2 \operatorname{cis}\theta$

$\therefore \operatorname{cis}\theta = \frac{z_2}{r_2}$

. generally well done

$\therefore$ from $z_1 = r_1 \operatorname{cis}\theta = r_1 \cdot \frac{z_2}{r_2}$

$\therefore z_1 = k z_2$ where $k = \frac{r_1}{r_2}$

① deducing relationship.

---

ii) Side AB is either $\overrightarrow{AB}$ or $\overrightarrow{BA}$

$OA + AB = OB$ $\quad OB + BA = OA$

$\therefore AB = OB - OA = z_2 - z_1$

$\therefore BA = OA - OB = z_1 - z_2 = -(z_2 - z_1)$

① for both possibilities

. ok

iii) side CD is given by $z_3 - z_4$ or $-(z_4 - z_3)$

as CD ∥ AB, $(z_4 - z_3) = k(z_2 - z_1)$

but opposite sides of a parallelogram are equal,

so $|z_4 - z_3| = |z_2 - z_1| \implies k = \pm 1$

$\underline{k=1}$: $\therefore z_4 - z_3 = z_2 - z_1$

or $z_1 - z_2 - z_3 + z_4 = 0$

$\underline{k=-1}$: $\therefore z_4 - z_3 = -(z_2 - z_1)$

but from (ii), AB (or BA) can be either $z_2 - z_1$ or $-(z_2 - z_1)$

$\therefore z_4 - z_3 = z_2 - z_1$

$\therefore z_1 - z_2 - z_3 + z_4 = 0$

*Marking:* ① side CD; ① deriving k; ① both cases for k.

*Comments:* · not well done.

iv) the argument uses only real numbers, not complex numbers, thus

$-1 = (a+ib)^2$, where a, b are real.

$= a^2 - b^2 + 2abi$

$\therefore a^2 - b^2 = -1$ and $2ab = 0$

$b = 0 \implies a^2 = -1$, which cannot happen as a is real

so $a = 0$ and $-b^2 = -1 \implies b = \pm 1$

$\therefore$ the roots are $-i$ and $i$

$i^2 = -1$ and $(-i)^2 = (-1)^2 i^2 = -1$

thus $\sqrt{-1} \times \sqrt{-1} = \sqrt{1 \times -1} \times \sqrt{1 \times -1} = 1i \times 1i \neq \sqrt{(-1)\times(-1)}$.

*Marking:* ① identifies use of real nos to try and solve a complex no problem; ① demonstrates correct procedure (in some way).

*Comments:* · Some good answers, most realized something was wrong, but couldn't articulate what it was

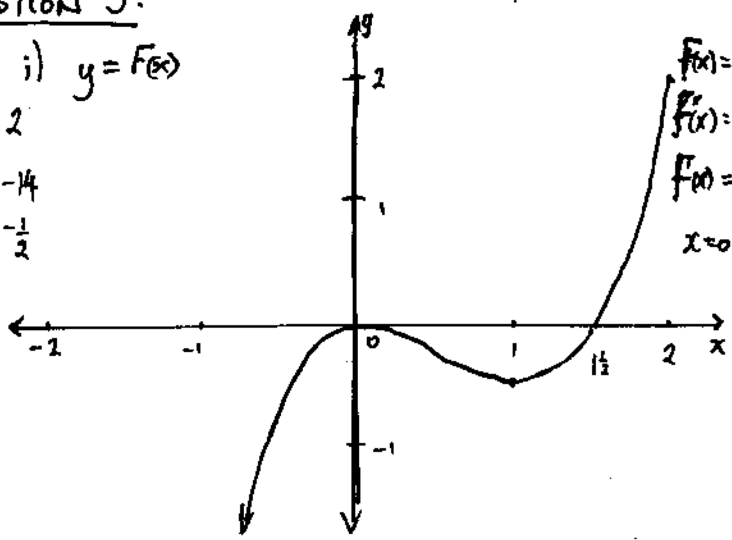
QUESTION 3:

a) i) $y = f(x)$

$f(2) = 2$

$f(-2) = -14$

$f(1) = -\frac{1}{2}$



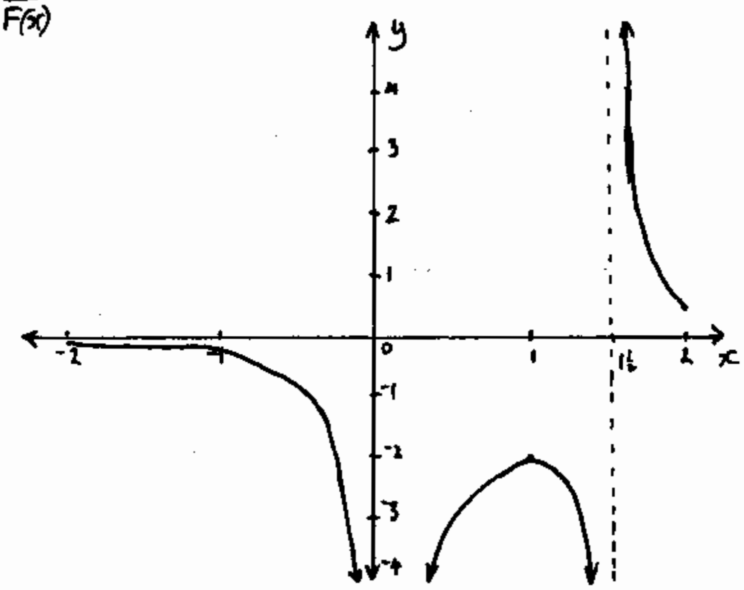

$f(x) = x^3 - \frac{3}{2}x^2$

$f'(x) = 3x^2 - 3x$

$f'(x) = 0$ gives $x = 0$, $x = 1$

*Marking:* ① axis intercepts; ① turning pt. @ $(1, -\frac{1}{2})$

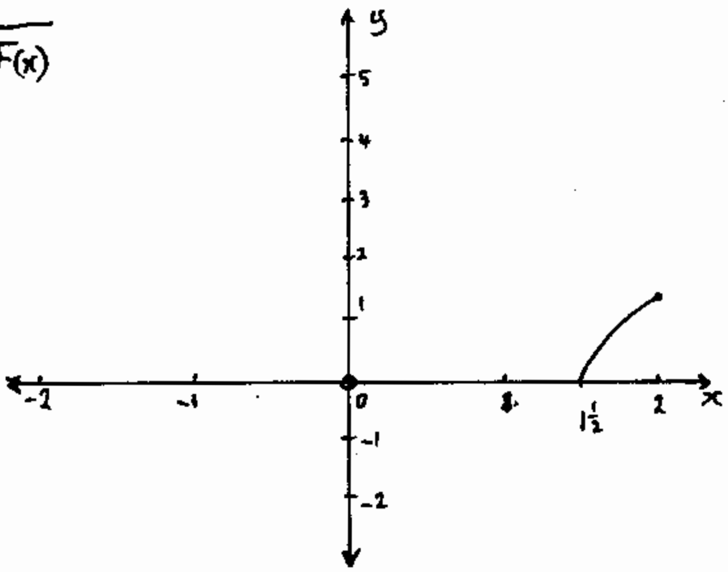
$y = \frac{1}{F(x)}$



① asymtotes

① correct shape in areas.

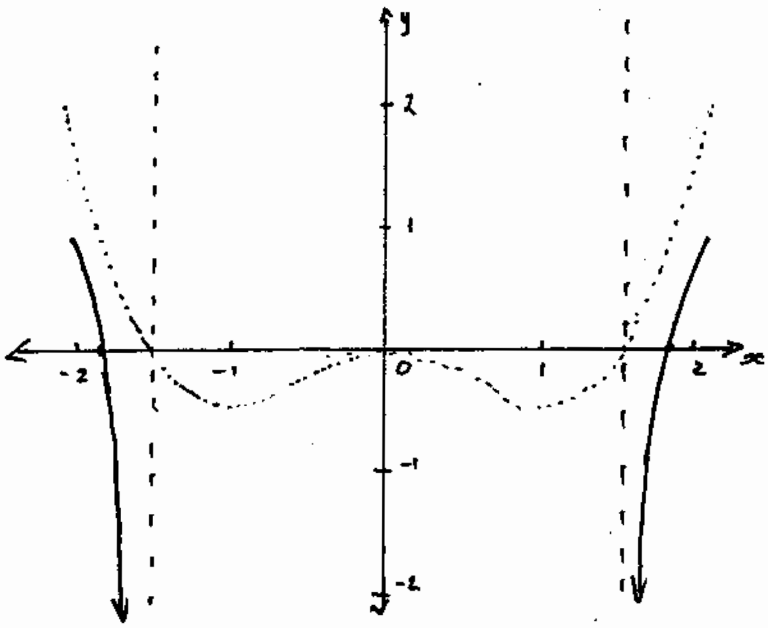
i) $y = \sqrt{F(x)}$



① correct intercept

· no penalty for (0,0) not plotted

① correct shape in areas



① asymptotes

① correct shape in area.

OLUTIONS: Yr 12 TRIAL HSC. EXTN II: 2003 | MARKING | COMMENTS

)i) $a = 5$, $b = 3$ and for hyperbola: $b^2 = a^2(e^2 - 1)$

$$\therefore 9 = 25(e^2 - 1)$$
$$e^2 - 1 = \frac{9}{25}$$
$$e^2 = \frac{34}{25}$$
$$\therefore e = \frac{\sqrt{34}}{5}$$

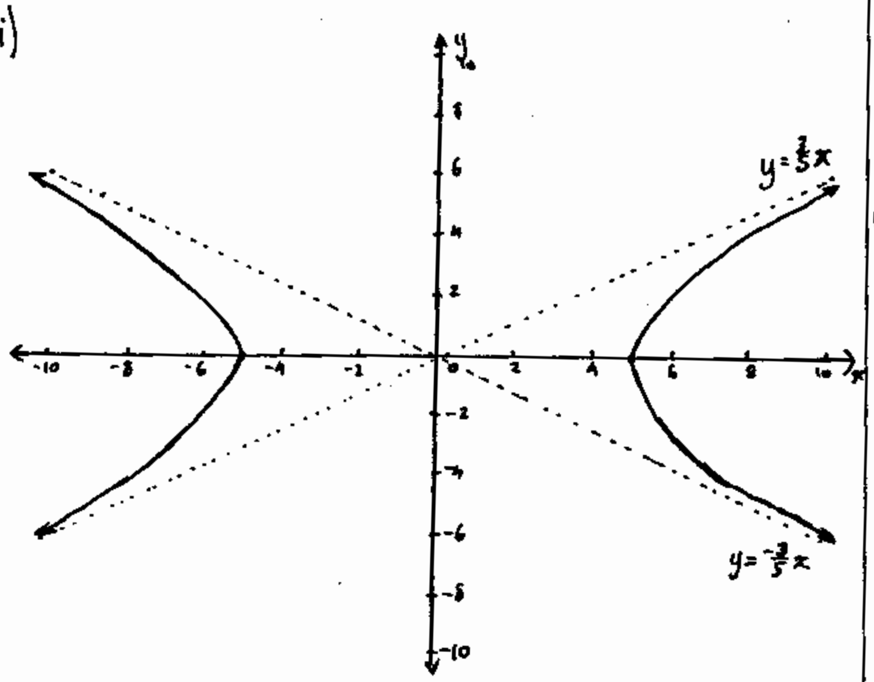
① correct value of e

ii) Foci are $S(ae, 0)$ and $S'(-ae, 0)$

$$\therefore (\sqrt{34}, 0) \text{ and } (-\sqrt{34}, 0)$$

① correct foci.

iii)




① asymptotes

① correct shape + intercepts.

• could also get this mark if shape is reasonable and scale indicated on y-axis

) $x = 6$: $\frac{36}{25} - \frac{y^2}{9} = 1$

$$\frac{36}{25} - 1 = \frac{y^2}{9}$$
$$y^2 = \frac{9 \times 11}{25}$$
$$\therefore y = \pm \frac{\sqrt{99}}{5}$$

$\therefore$ A is $\left(6, \frac{\sqrt{99}}{5}\right)$ and B is $\left(6, -\frac{\sqrt{99}}{5}\right)$

① correct A and B

) $\frac{x^2}{25} - \frac{y^2}{9} = 1$

$$\therefore \frac{2x}{25} - \frac{2y}{9} \cdot \frac{dy}{dx} = 0$$
$$\text{so } \frac{dy}{dx} = \frac{-2x}{25} \cdot \frac{-9}{2y}$$
$$= \frac{9x}{25y}$$

at $\left(6, \frac{\sqrt{99}}{5}\right)$, $\frac{dy}{dx} = \frac{9 \cdot 6 \cdot 5}{25 \cdot \sqrt{99}}$

$$= \frac{54}{5\sqrt{99}}$$

① correct differentiation for $\frac{dy}{dx}$

① correct subst to eqn.

• ignored small arithmetic errors

(cont) $\therefore\ y - \frac{\sqrt{99}}{5} = \frac{54}{5\sqrt{99}}(x-6)$

$$5\sqrt{99}\,y - 99 = 54x - 324$$

$$\therefore\ 0 = 54x - 5\sqrt{99}\,y - 225$$

UESTION 4:

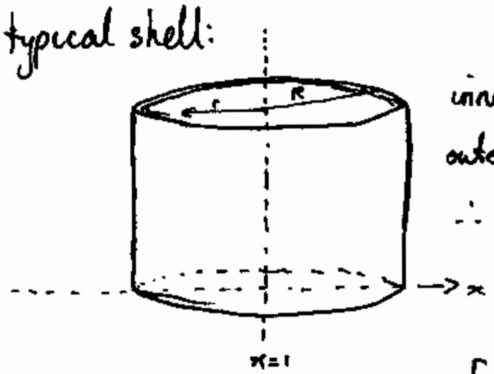
a) i) at $P$: $3-x^2 = x^2 - x$

$$\therefore\ 0 = 2x^2 - x - 3$$
$$= 2x^2 + 2x - 3x - 3$$
$$= 2x(x+1) - 3(x+1)$$
$$= (x+1)(2x-3)$$
$$\therefore\ x = -1,\ 3/2$$

$\therefore$ $x$ co-ord of $P$ is $-1$ (as $P$ is in 2nd quadrant)

① correct value for $x$.

) typical shell:




inner radius: $r = 1-x$

outer Radius: $R = 1-(x+\delta x)$

$\therefore$ Area of annulus:

$$\delta A = \pi R^2 - \pi r^2$$
$$= \pi\left(1-(x+\delta x)\right)^2 - \pi(1-x)^2$$
$$= \pi\left[1-2(x+\delta x)+(x+\delta x)^2-(1-2x+x^2)\right]$$
$$= \pi\left[1-2x-2\delta x+x^2+2x\delta x+\delta x^2-1+2x-x^2\right]$$
$$= \pi\left(2x\delta x - 2\delta x + \delta x^2\right)$$
$$= 2\pi(x-1)\delta x \quad \text{(ignoring } \delta x^2 \text{ as too small).}$$

① area of annulus $\delta A$

$\therefore$ a small volume of shell is given by

$\delta V = \delta A . h$ where $h = (3-x^2)-(x^2-x)$
$$= 3 + x - 2x^2$$

① correct $h$ leading to $\delta V$

alternatively:
1 mark: volume of typical shell
1 mark: correct limits
1 mark: integration

$$\therefore\ \delta V = 2\pi(x-1)(3+x-2x^2)\delta x$$
$$= 2\pi(3x + x^2 - 2x^3 - 3 - x + 2x^2)\delta x$$
$$= 2\pi(-3+2x+3x^2-2x^3)\delta x$$

$\therefore$ Volume of the solid is given by

$$V = \sum \delta V$$
$$= \lim_{\delta x \to 0} \sum_{x=-1}^{1} 2\pi(-3+2x+3x^2-2x^3)\delta x$$
$$= 2\pi\int_{-1}^{1} -3+2x+3x^2-2x^3\,dx$$

① correct summing leading to integral.

| Solution | Marking | Comments |
|---|---|---|
| ii) $\therefore V = 2\pi\left\lvert\left[-3x + x^2 + x^3 - \tfrac{1}{2}x^4\right]_{-1}^{1}\right\rvert$ <br> $= 2\pi\left\lvert\left[(-3+1+1-\tfrac{1}{2}) - (3+1-1-\tfrac{1}{2})\right]\right\rvert$ <br> $= 2\pi\left\lvert -1\tfrac{1}{2} - 2\tfrac{1}{2}\right\rvert$ <br> $= 8\pi$ | ① correct integration <br> ① correct answer | • Full marks only for correct solution. |
| ) $\therefore \dfrac{x}{(x-1)^2(x-2)} = \dfrac{A(x-1)(x-2) + B(x-2) + C(x-1)^2}{(x-1)^2(x-2)}$ <br> $\therefore\ x = A(x-1)(x-2) + B(x-2) + C(x-1)^2$ — ① <br> in ①: $x=1$ gives: $1 = B(1-2)$ ie $B=-1$ <br> $x=2$ gives: $2 = C(2-1)^2$ ie $C=2$ <br> also, from ①: $x = A(x^2-3x+2) + B(x-2) + C(x^2-2x+1)$ <br> equating co-efficients of $x^2$: <br> $0 = A + C$ <br> $\therefore\ A = -2$ <br> $\therefore \dfrac{x}{(x-1)^2(x-2)} = \dfrac{-2}{(x-1)} - \dfrac{1}{(x+1)^2} + \dfrac{2}{x-2}$ | ① subst to find B, C (or any other method) <br> ① equating co-effs to find A. | |
| $\therefore \displaystyle\int_0^{1/2} \frac{x}{(x-1)^2(x-2)}\,dx$ <br> $= \displaystyle\int_0^{1/2} \frac{2}{(x-2)} - \frac{2}{x-1} - \frac{1}{(x-1)^2}\,dx$ <br> $= \displaystyle\int_0^{1/2} \frac{-2}{2-x} + \frac{2}{1-x} - \frac{1}{(x-1)^2}\,dx$ | ① correct rearrangement to get to integration | |
| $= \left[-2\ln(2-x).(-1) + 2\ln(1-x).(-1) + \dfrac{1}{x-1}\right]_0^{1/2}$ <br> $= \left[2\ln(2-x) - 2\ln(1-x) + \dfrac{1}{x-1}\right]_0^{1/2}$ <br> $= 2\ln\tfrac{3}{2} + 2\ln\tfrac{1}{2} - 2 - (2\ln 2 - 2\ln 1 - 1)$ <br> $= 2\ln\tfrac{3}{2} + 2\ln 2 - 2 - 2\ln 2 + 0 + 1$ <br> $= 2\ln\left(\tfrac{3}{2}\right) - 1.$ | ① correct subst to show answer. | |
| c) $\cos 2x = c = \sin 3x$ $c$ constant <br> $\therefore\ \sin 3x = c$ <br> or $\cos\left(\tfrac{\pi}{2} - 3x\right) = c$ <br> $\tfrac{\pi}{2} - 3x = \cos^{-1}(c) + 2\pi n$ $n = 0, \pm 1, \pm 2 \ldots$ | | |

) (cont) $\therefore \quad -3x = \frac{-\pi}{2} + 2\pi n + \cos^{-1}(c)$

or $\quad 3x = \frac{\pi}{2} - 2\pi n - \cos^{-1}(c)$

① correct setup of problem (any method)

but $\cos 2x = c$ also,

so $\quad 2x = \cos^{-1}(c)$

$$\therefore \quad 3x = \frac{\pi}{2} - 2\pi n - 2x$$

$$5x = \left(\frac{1-4n}{2}\right)\pi$$

$$\therefore x = \left(\frac{1-4n}{10}\right)\pi$$

① correct solutions in range.

for $0 \le x \le \frac{\pi}{2}$, we get (using $n=0, n=-1$)

$$x = \frac{\pi}{10}, \frac{\pi}{2}$$

---

d) let $\tan^{-1} 3x = \theta$ and $\tan^{-1} 2x = \phi$

$\therefore \tan\theta = 3x \qquad \tan\phi = 2x$

for $\tan^{-1}\left(\frac{1}{5}\right) = \tan^{-1}(3x) - \tan^{-1}(2x)$

$= \theta - \phi$

① correct use of tan

taking tan of both sides:

$$\tan\left(\tan^{-1}\frac{1}{5}\right) = \tan(\theta - \phi)$$

$$\therefore \frac{1}{5} = \frac{\tan\theta - \tan\phi}{1 + \tan\theta\tan\phi}$$

$$= \frac{3x - 2x}{1 + 3x.2x}$$

① forms quadratic

$$1 + 6x^2 = 5x$$

$$\text{or} \quad 0 = 6x^2 - 5x + 1$$

$$= 6x^2 - 3x - 2x + 1$$

$$= 3x(2x-1) - 1(2x-1)$$

$$= (2x-1)(3x-1)$$

$$\therefore x = \frac{1}{2}, \frac{1}{3}$$

① correct answers

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QUESTION 5:

a) $\alpha + \beta + \gamma = -4$

$\alpha\beta + \alpha\gamma + \beta\gamma = 2$

$\alpha\beta\gamma = 3$

i) $\alpha^2 + \beta^2 + \gamma^2 = (\alpha + \beta + \gamma)^2 - 2(\alpha\beta + \beta\gamma + \alpha\gamma)$

$= (-4)^2 - 2(2)$

$= 12$

① answer

• some had an incorrect squares expansion!

---

ii) for roots $x = 1-\alpha \Rightarrow \alpha = (1-x)$

$\therefore (1-x)$ in eqn gives:

$$(1-x)^3 + 4(1-x)^2 + 2(1-x) - 3 = 0$$

$$1 - 3x + 3x^2 - x^3 + 4 - 8x + 4x^2 + 2 - 2x - 3 = 0$$

$$\therefore \quad 4 - 13x + 7x^2 - x^3 = 0$$

$$\text{or} \quad x^3 - 7x^2 + 13x - 4 = 0$$

① correct setup with roots
① correct eqn.

· many simple algebraic errors

iii) for roots $x = \frac{1}{\alpha} \Rightarrow \alpha = \frac{1}{x}$

$$\therefore \left(\tfrac{1}{x}\right)^3 + 4\left(\tfrac{1}{x}\right)^2 + 2\left(\tfrac{1}{x}\right) - 3 = 0$$

$\times x^3$:

$$1 + 4x + 2x^2 - 3x^3 = 0$$

$$\text{or} \quad 3x^3 - 2x^2 - 4x - 1 = 0$$

① setup with roots
① correct eqn.

b) let $\alpha$ be the root of multiplicity 3, then $P(\alpha) = P'(\alpha) = P''(\alpha) = 0$.

$$\therefore P'(x) = 32x^3 - 75x^2 + 54x - 11$$

$$P''(x) = 96x^2 - 150x + 54$$

if $P''(\alpha) = 0$, $\alpha$ is the soln to $0 = 96x^2 - 150x + 54$

$$\text{or} \quad 0 = 48x^2 - 75x + 27$$

$$\therefore x = \frac{75 \pm \sqrt{2625 - 4.48.27}}{96}$$

$$= \frac{75 \pm \sqrt{441}}{96}$$

$$= \frac{75 \pm 21}{96}$$

$$= 1,\ \frac{9}{16}$$

now, $P'(1) = 32 - 75 + 54 - 11$

$= 0$

and $P(1) = 8 - 25 + 27 - 11 + 1$

$= 0$

$\therefore (x-1)^3$ is a factor of $P(x)$

so $\alpha = 1$ is the tripple root.

Also, $\alpha^3 . \beta = \frac{1}{8}$

$\therefore \beta = \frac{1}{8}$ is the other root.

① set up problem with $P''(\alpha) = 0$.
① correct possibilities for tripple root.
① correct tripple root with reasons
① other root

· several used $P'''(\alpha) = 0$
· many did not understand the implications for a root with multiplicity!
· need to explicitly state the other root. $(8x-1)$ as a factor implies a root of $x = \frac{1}{8}$.

c) let $\alpha, \beta, \gamma$ and $\delta$ be the roots of the equation

i) $\therefore x = \alpha + 1$ is a root of the reqd eqn

so $\alpha = x - 1$

$$\therefore (x-1)^4 + 4(x-1)^3 + 5(x-1)^2 + 2(x-1) - 20 = 0$$

① correct subst for root.

| Solution | MARKING | COMMENTS |
|---|---|---|
| (cont): $(x^4-4x^3+6x^2-4x+1)+(4x^3-12x^2+12x-4)+(5x^2-10x+5)+2x-2-20=0$ <br> $\therefore x^4-x^2-20=0$ as reqd. | ① correct alg. to soln. | • many errors expanding this |
| i) now $x^4-x^2-20=0$ <br> $(x^2-5)(x^2+4)=0$ <br> $\therefore x^2=5,-4$ <br> $\therefore x=\pm\sqrt5,\ \pm 2i$ <br> $\therefore$ the roots of $x^4+4x^3+5x^2+2x-20=0$ are given by $\alpha=x-1$ <br> $\therefore$ roots are $-1+\sqrt5, -1-\sqrt5, -1+2i, -1-2i$ | ① correct roots for $x^4-x^2-20=0$ <br> ① correct roots for orig. eqn. | • many had trouble linking the roots back to the original with $\alpha=x-1$ |
| QUESTION 6: <br> a) i) $x_p=OS$ $=5\cos\theta$ $\quad y_p=OT$ $=3\sin\theta$ | ① <br> ① each answer | • link back to the definition given in the diagram. This is the starting point, and many missed it. |
| ii) $\therefore \frac{x}{5}=\cos\theta$ and $\frac{y}{3}=\sin\theta$ <br> $\frac{x^2}{25}=\cos^2\theta$ $\quad \frac{y^2}{9}=\sin^2\theta$ <br> $\therefore \frac{x^2}{25}+\frac{y^2}{9}=\cos^2\theta+\sin^2\theta$ <br> so $\frac{x^2}{25}+\frac{y^2}{9}=1$ is the cartesian eqn. | ① correct eqn. | • "eliminate $\theta$" $\Rightarrow$ show how this happens, don't just write the equation down! |
| iii) normal to an ellipse is given by $\frac{ax}{\cos\theta}-\frac{by}{\sin\theta}=a^2-b^2$ where $a=5, b=3$, $\theta=\frac{\pi}{3}$ <br> $\therefore \frac{5x}{\cos\frac{\pi}{3}}-\frac{3y}{\sin\frac{\pi}{3}}=25-9$ <br> $10x-\frac{6y}{\sqrt3}=16$ <br> $\therefore 10\sqrt3x-6y-16\sqrt3=0$ is eqn. | ① correct subst in formula <br> ① correct eqn (any form) | • many found tangent instead of normal. <br> • put your answer into one of the standard simplest forms - many left their answer unfinished |
| b) i) $S'(x)=\frac{d}{dx}\left(\frac12(e^x-e^{-x})\right)$ <br> $=\frac12(e^x+e^{-x})$ <br> $=C(x)$ | ① set out clearly. | |
| ii) $e^x>0$ for all $x$ <br> $e^{-x}>0$ for all $x$ <br> $\therefore e^x+e^{-x}>0$ for all $x$ <br> $\therefore C(x)>0$ for all $x$ <br> ie $S'(x)>0$ for all $x$ <br> $\Rightarrow S(x)$ is monotonically increasing | ① correct reasoning. | • asked to show $\Rightarrow$ give reasons why $S'(x)>0$. Just stating it earns no marks. |

iii)

$$[C(x)]^2 = \left[\tfrac{1}{2}(e^x+e^{-x})\right]^2$$
$$= \tfrac{1}{4}(e^{2x}+2e^xe^{-x}+e^{-2x})$$
$$= \tfrac{1}{4}(e^{2x}+e^{-2x}+2)$$
$$1+[S(x)]^2 = 1+\left[\tfrac{1}{2}(e^x-e^{-x})\right]^2$$
$$= 1+\tfrac{1}{4}(e^{2x}-2e^xe^{-x}+e^{-2x})$$
$$= \tfrac{1}{4}(4+e^{2x}-2+e^{-2x})$$
$$= \tfrac{1}{4}(e^{2x}+e^{-2x}-2)$$
$$= [C(x)]^2 \text{ from above}$$

Marking: (1) expression for $[C(x)]^2$ correct. (1) reduction of $1+[S(x)]^2$ correct (or equivalent)

iv) as $S(x)$ is monotonically increasing, each $x$ must produce a unique $y$ value
$\Rightarrow S(x)$ has a 1-1 correspondence
$\therefore S^{-1}(x)$ exists for all values of $x$.

Marking: (1) appropriate explaination

Comments: • many attempted explainations revealed a lack of understand… of what inverse *means*.

v)

$$y = S^{-1}(x)$$
$$\therefore S(y) = x$$
$$\therefore \frac{dx}{dy} = S'(y)$$
$$= C(y)$$
$$= \sqrt{1+[S(y)]^2}$$
$$= \sqrt{1+x^2}$$
$$\therefore \frac{dy}{dx} = \frac{1}{\sqrt{1+x^2}}$$

Marking: (1) inverse rules to give $\frac{dx}{dy}$ correctly. (1) correct subst to formula.

Comments: • very few picked up the links to $S(x)$ and $C(x)$ in the previou… parts, so many futile atte… at a simple problem. Look at linked parts like this one - th… make the solution simpler!

vi)

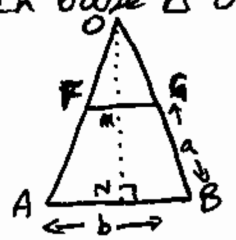
$$\therefore y = \int\frac{dx}{\sqrt{1+x^2}} \qquad \text{let } x=\tan\theta \quad \therefore dx = \sec^2\theta\, d\theta$$
$$= \int\frac{\sec^2\theta\, d\theta}{\sqrt{1+\tan^2\theta}}$$
$$= \int\frac{\sec^2\theta\, d\theta}{\sec\theta}$$
$$= \int\sec\theta\, d\theta$$
$$= \int\frac{\sec\theta(\sec\theta+\tan\theta)}{(\sec\theta+\tan\theta)}\, d\theta$$
$$= \ln(\sec\theta+\tan\theta)+c$$
$$\therefore y = \ln(x+\sqrt{1+x^2})+c$$


Marking: (1) reduction to $\int\sec\theta\, d\theta$. (1) correct $\int$ of $\sec\theta$. (1) correct subst to give $y$ in terms of $x$.

Comments: • The question is to *show* the relationship $\Rightarrow$ *not* using the standard integral table. The integration *is* the question, its not part of something bigger.

QUESTION 7:

a) i) In base $\Delta$ OAB:




$GB = a \qquad OB = r$

$NB = \frac{b}{2} \qquad OG = r-a$

Marking: (1) correct setup of variables

i) (cont) $\therefore \dfrac{MG}{OG} = \dfrac{NB}{OB}$

or $MG = \dfrac{NB \cdot OG}{OB}$

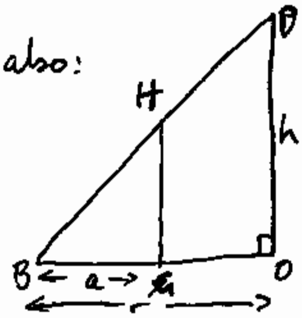
$= \dfrac{b}{2} \cdot \dfrac{(r-a)}{r}$

$\therefore FG = 2MG$

$= \dfrac{b(r-a)}{r}$

① correct value for FG

also:




$OD = h, \quad OB = r$

$GB = a$

$\therefore \dfrac{GH}{GB} = \dfrac{OD}{OB}$

$\therefore GH = \dfrac{OD \cdot GB}{OB}$

$= \dfrac{a \cdot h}{r}$

① correct expression for GH.

$\therefore V_s = FG \cdot GH \cdot \delta a$

$= \left(\dfrac{r-a}{r}\right) b \cdot \left(\dfrac{ah}{r}\right) \delta a$

---

ii) $\therefore V = \displaystyle\int_0^r \left(\frac{r-a}{r}\right) b \cdot \left(\frac{ah}{r}\right) da$

$= \dfrac{bh}{r^2} \displaystyle\int_0^r a(r-a)\, da$

$= \dfrac{bh}{r^2} \displaystyle\int_0^r ar - a^2\, da$

$= \dfrac{bh}{r^2} \left[\frac{1}{2}a^2 r - \frac{1}{3}a^3\right]_0^r$

$= \dfrac{bh}{r^2} \left[\left(\frac{1}{2}r^3 - \frac{1}{3}r^3\right) - 0\right]$

$= \dfrac{bh}{\cancel{r^2}} \cdot \left(\frac{1}{2} - \frac{1}{3}\right) r^{\cancel{3}}$

$= \dfrac{1}{6} bhr$ as reqd.

① reduction to correct $\int$

① correct subst to expression

---

iii) given $\angle AOB = \dfrac{2\pi}{n}$


ie $\theta = \dfrac{2\pi}{n}$

$\therefore \dfrac{\theta}{2} = \dfrac{\pi}{n}$

$\therefore \sin\dfrac{\theta}{2} = \dfrac{b}{2} \cdot \dfrac{1}{r}$

or $b = 2r \sin\dfrac{\theta}{2}$

$= 2r \sin\dfrac{\pi}{n}$

① correct derivation for b.

$\therefore V = \dfrac{1}{6} bhr$ from (ii) above

so $V = \dfrac{1}{6} hr \cdot 2r \sin\dfrac{\pi}{n}$

$= \dfrac{1}{3} hr^2 \sin\dfrac{\pi}{n}$

ie $V_n = \dfrac{1}{3} hr^2 n \sin\dfrac{\pi}{n}$

① correct expression for V.

(iii) (cont) $\therefore \lim_{n\to\infty} V_n = \lim_{n\to\infty} \frac{1}{3} n h r^2 \sin\frac{\pi}{n}$

$= \lim_{n\to\infty} \frac{1}{3} h r^2 \pi . \dfrac{\sin\frac{\pi}{n}}{\frac{\pi}{n}}$

let $x = \frac{\pi}{n}$; as $n \to \infty$ $\frac{\pi}{n} \to 0$

$\therefore \lim_{n\to\infty} V_n = \lim_{n\to\infty} \frac{1}{3} h r^2 \pi . \lim_{x\to 0} \frac{\sin x}{x}$

$= \frac{1}{3}\pi r^2 h$

**Marking:** ① limit expression; ① correct use of limits to evaluate expression

**Comments:** · many incorrect uses of the limit.

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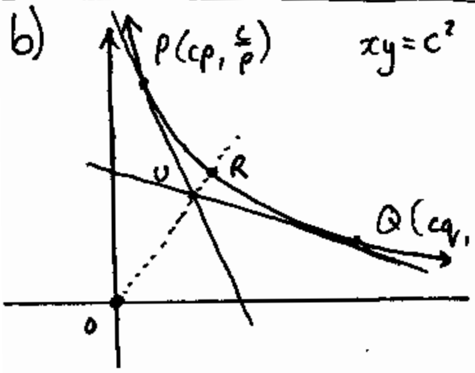
b) $xy = c^2$ - rectangular hyperbola




**Marking:** ① diagram showing relationships

**Comments:** · many students didn't draw correct diagrams!

tangent at P: $x + p^2 y = 2cp$ — ①

" Q: $x + q^2 y = 2cq$ — ②

$(p^2 - q^2) y = 2c(p - q)$ : ① - ②

$\therefore y_U = \frac{2c}{p+q}$

$\therefore x_U = 2cp - p^2 \frac{2c}{p+q}$

$= \frac{2cpq}{p+q}$

Now $m_{OU} = \frac{2c}{p+q} . \frac{p+q}{2cpq}$

$= \frac{1}{pq}$

$\therefore$ eqn of OUR is $y = \frac{x}{pq}$ — ③

$xy = c^2$ — ④

subst ③ in ④: $\frac{x^2}{pq} = c^2$

$\therefore x^2 = pqc^2$

$\therefore x_R = c\sqrt{pq}$

but R is $(cr, \frac{c}{r})$

so $cr = c\sqrt{pq}$

$r = \sqrt{pq}$

or $r^2 = pq$

**Marking:** ① finding y co-ord of U; ① finding x co-ord of U; ① gradient OU; ① finding $x_R$ (or $y_R$); ① correct relationships (either form)

**Comments:** · not many students completed the correct relationship. · Some became lost after getting correct points of intersection, others made it much more complicated than it was.

QUESTION 8:

a) i) $\int_0^{2/3} \sqrt{4-9x^2}\,dx$

$x = \frac{2}{3}\sin\theta$
$\therefore dx = \frac{2}{3}\cos\theta\,d\theta$
when $x = \frac{2}{3}$, $\theta = \frac{\pi}{2}$
$x = \theta$, $\theta = 0$

$\therefore = \int_0^{\frac{\pi}{2}} \sqrt{4-4\sin^2\theta} \cdot \frac{2}{3}\cos\theta\,d\theta$

① correct subst to $\theta$, including limits

$= \frac{4}{3}\int_0^{\frac{\pi}{2}} \sqrt{\cos^2\theta}\,.\cos\theta\,d\theta$

① correct reduction to $\int \cos^2$

$= \frac{4}{3}\int_0^{\frac{\pi}{2}} \cos^2\theta\,d\theta$

$= \frac{4}{3}\int_0^{\frac{\pi}{2}} \frac{1}{2}(\cos 2\theta + 1)\,d\theta$

① correct subst to soln.

$= \frac{2}{3}\left[\frac{1}{2}\sin 2\theta + \theta\right]_0^{\frac{\pi}{2}}$

$= \frac{2}{3}\left[\left(\frac{1}{2}\cdot 0 + \frac{\pi}{2}\right) - 0\right]$

$= \frac{\pi}{3}$

ii) for $9x^2 + y^2 = 4$

$y^2 = 4 - 9x^2$

$\therefore y = \sqrt{4-9x^2}$

$\therefore$ pt i) gives the area in the first quadrant so, from symmetry, this is $\frac{1}{4}$ the reqd. area.

$\therefore A = 4 \cdot \frac{\pi}{3}$

$= \frac{4\pi}{3}$ sq units.

① answer.

b) i) $x = a\sin\theta$ $\therefore dx = a\cos\theta\,d\theta$

when $x = a$, $\theta = \frac{\pi}{2}$
+ $x = 0$, $\theta = 0$

$\therefore \int_0^a \sqrt{a^2 - x^2}\,dx$

$= \int_0^{\frac{\pi}{2}} \sqrt{a^2 - a^2\sin^2\theta}\,.a\cos\theta\,d\theta$

① correct subst inc limits

$= a^2\int_0^{\frac{\pi}{2}} \sqrt{1-\sin^2\theta}\,.\cos\theta\,d\theta$

$= a^2\int_0^{\frac{\pi}{2}} \cos^2\theta\,d\theta$

① correct integration to solution

$= a^2\int_0^{\frac{\pi}{2}} \frac{1}{2}(\cos 2\theta - 1)\,d\theta$

$= a^2\left[\frac{1}{4}\sin 2\theta - \theta/2\right]_0^{\pi/2}$

$= a^2\left[\left(0 + \frac{\pi}{4}\right) - 0\right]$

$= \frac{a^2\pi}{4}$

| Solution | Marking | Comments |
|---|---|---|
| ii) from $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$<br>$b^2x^2+a^2y^2=a^2b^2$<br>$a^2y^2=a^2b^2-b^2x^2$<br>$\therefore y^2=b^2-\frac{b^2}{a^2}x^2$<br>$\therefore y=\sqrt{b^2-\frac{b^2}{a^2}x^2}$<br>$=\sqrt{\frac{b^2a^2-b^2x^2}{a^2}}$<br>$=\frac{b}{a}\sqrt{a^2-x^2}$ | ① reducing equation to stnd. form | |
| $\therefore$ area of 1st quadrant is<br>$A_1=\int_0^a \frac{b}{a}\sqrt{a^2-x^2}\,dx$<br>$=\frac{b}{a}\cdot\frac{\pi a^2}{4}$ from (i)<br>$=\frac{\pi ab}{4}$<br>$\therefore$ total area (from symmetry)<br>$A=4\cdot\frac{\pi ab}{4}$<br>$=\pi ab$ | ① correct reasoning to soln. | |
| c) $\Delta V=A\,\Delta h$ where $A$ is the area of the ellipse at height $h$.<br>$\therefore$ from (b) above:<br>$A=\pi ab\lambda^2$<br>(as $\frac{x^2}{a^2}+\frac{y^2}{b^2}=\lambda^2$ becomes $\frac{x^2}{a^2\lambda^2}+\frac{y^2}{b^2\lambda^2}=1$ ) | ① correct deduction of A | |
| $\therefore \Delta V=\pi ab\lambda^2\Delta h$<br>$\therefore V=\int_0^H \pi ab\left(1-\frac{h^2}{H^2}\right)^2 dh$<br>$=\pi ab\int_0^H 1-\frac{2h^2}{H^2}+\frac{h^4}{H^4}\,dh$ | ① correct expression for V in terms of h's. | . wrong $\lambda$, but correct method, gained 1 mark |
| $=\pi ab\left[h-\frac{2}{3}\frac{h^3}{H^2}+\frac{h^5}{5H^4}\right]_0^H$<br>$=\pi ab\left[\left(H-\frac{2}{3}\frac{H^3}{H^2}+\frac{H^5}{5H^4}\right)-0\right]$<br>$=\pi ab\left[\frac{15H-10H+3H}{15}\right]$<br>$=\frac{8\pi abH}{15}$ as reqd. | ① correct $\int$ leading to soln | |

d) $x^2-(2\cos\theta)x+1=0$

i) $x^2-(2\cos\theta)x+\cos^2\theta=-1+\cos^2\theta$

$\therefore\ (x-\cos\theta)^2=-\sin^2\theta$

$\therefore\ x-\cos\theta=\pm i\sin\theta$

$\therefore\ x=\cos\theta\pm i\sin\theta$

$\therefore\ \alpha=\cos\theta+i\sin\theta \quad \beta=\cos\theta-i\sin\theta$

*Marking:* both ① answers

ii) $\alpha=\text{cis}\,\theta$

$\therefore\ \alpha^{10}=(\text{cis}\,\theta)^{10}$

$=\text{cis}\,10\theta$ by deMoivres Theorem

similarly $\beta=\overline{\text{cis}\,\theta}$

so $\beta^{10}=(\overline{\text{cis}\,\theta})^{10}$

$=\overline{\text{cis}\,10\theta}$

$\therefore\ \alpha^{10}+\beta^{10}=\text{cis}\,10\theta+\overline{\text{cis}\,10\theta}$

$=\cos 10\theta+i\sin 10\theta+\cos 10\theta-i\sin 10\theta$

$=2\cos 10\theta$ as reqd.

*Marking:* ① correct use of deMoivres Theorem; ① correct use of $\overline{\text{cis}\,\theta}$; ① correct algebra to soln.