Moriah College

HSC Trial Examination

Mathematics Extension 2

2002

1. **(a)** Find $\int \frac{1}{\sqrt{4x-x^2}}\, dx$

(b) Find the exact value of k if $\int_1^{k^2} \frac{e^{\sqrt{x}}}{\sqrt{x}}\, dx = 6e$

(c) Let $I_n = \int_1^e x(\ln x)^n\, dx$ where n is a positive integer.

(i) Use integration by parts to show that $I_n = \frac{e^2}{2} - \frac{n}{2} I_{n-1}$ for $n \geq 1$.

(ii) Hence find the value of $\int_1^e x(\ln x)^3\, dx$

(d) Find $\int_0^{\frac{\pi}{3}} \frac{1}{2+\cos 2x}\, dx$ using the substitution $u = \tan x$.

2. **(a)** Let $z = -\sqrt{3} + i$

(i) Write z in modulus-argument form.

(ii) Hence find z^8 in the form $x + iy$ where x and y are real.

(iii) Find the least positive integer value of n such that z^n is real.

(b) Sketch each of the following regions on separate diagrams.

(i) $-2 < \Re(z) < 1$ and $-1 < \Im(z) \leq 3$

(ii) $|z - 2 - i| \leq 2$

(iii) $0 < \arg[(1+i)z] \leq \frac{\pi}{2}$

(c) $OABC$ is a square. O represents 0, A represents $3 + i$, B represents z, C represents w and D is the point where the diagonals of the square meet.

(i) Find the complex numbers represented by C and D in the form $x + iy$.

(ii) Find the arg $\left(\frac{w}{z}\right)$

3. **(a)** **(i)** Find real numbers B and C such that $\frac{2x^2+5x}{(x-2)(x+3)} = 2 + \frac{B}{(x-2)} + \frac{C}{(x+3)}$

(ii) Hence find $\int \frac{2x^2+5x}{(x-2)(x+3)}\, dx$

(b) Consider the function given by $f(x) = \frac{2x^2+5x}{(x-2)(x+3)}$

(i) Noting that $f(-4) = 2$ sketch the graph of $y = f(x)$ showing all asymptotes and intercepts. There is no need for finding stationary points and inflexion points.

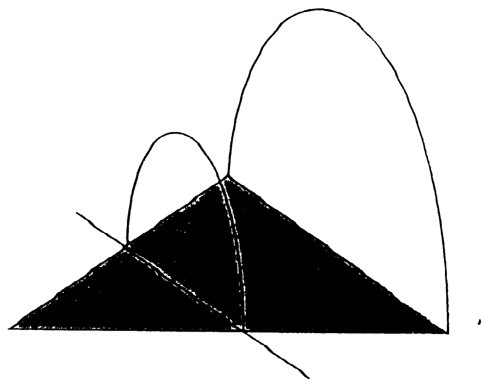
(ii) Hence sketch the graph of $g(x) = \frac{(x-2)(x+3)}{2x^2+5x}$ showing all asymptotes and intercepts.

(iii) By making x the subject of the expression $y = \frac{2x^2+5x}{(x-2)(x+3)}$ show that the range of the function is given by $\left\{y : y \leq \frac{29-6\sqrt{6}}{25} \ \vee \ y \geq \frac{29+6\sqrt{6}}{25}\right\}$.

(iv) Hence, or otherwise find the values of k such that the equation $\frac{2x^2+5x}{(x-2)(x+3)} = k$ hasno real solutions for x.

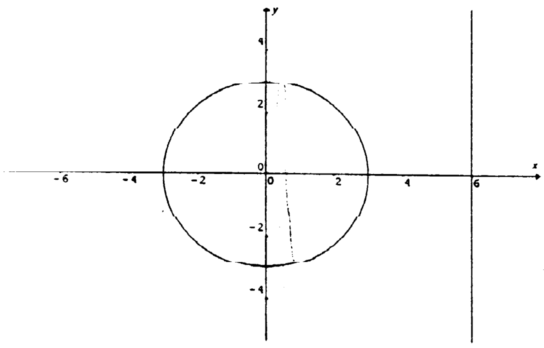
4. **(a)** Let $P(x) = ax^4 + bx^3 + cx^2 + dx + e$ where a, b, c, d and e are integers. If $x = \frac{1}{r}$ is a root of the polynomial equation $P(x) = 0$ then show that r is a factor of a.
(b) Consider the polynomial $P(x) = 8x^4 - 8x^2 - x + 1$. You are given that $P(x) = (x-1)Q(x)$ where $Q(x)$ is a cubic polynomial.
(i) Find $Q(x)$
(ii) If the equation $Q(x) = 0$ has ane rational root then find the it four solutions to the equations $P(x) = 0$
(c) **(i)** Solve the equation $\cos 4\theta = \cos\theta$ giving the general solution.
(ii) Show that $\cos 4\theta = 8\cos^4\theta - 8\cos^2\theta + 1$
(iii) Show that the equation $\cos 4\theta = \cos\theta$ can be expressed in the form $8x^4 - 8x^2 - x + 1 = 0$ where $x = \cos\theta$.
(iv) Hence using part **(b)** **(ii)** find the exact value of
(α) $\cos\frac{2\pi}{5}$ **(β)** $\cos\frac{\pi}{5}$

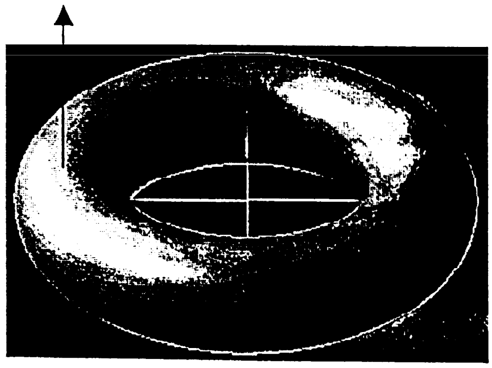
5. **(a)** Show that the area bounded by the parabola $x^2 = 4ay$ and the latus rectum $y = a$ is equal to $\frac{8a^2}{3}$.
(b) A particular solid has a triangular base with side lengths all 6 metres. Cross sections taken parallel to one side of the base are parabolas. Each parabolic cross-section is such that it has its latus rectum lying in the base of the solid. Using part **(a)** find the volume of the solid.



(c) **(i)** Show that the function $f(x) = x\sqrt{a^2 - x^2}$ is odd.

(ii) The circle with equation $x^2 + y^2 = 9$ is rotated about the line $x = 6$ to form a torus.





Show, using the method of cylindrical shells that the volume V of the torus is given by $V = 4\pi \int_{-3}^{3} (6 - x)\sqrt{9 - x^2}\, dx$

(iii) Hence find the volume of the torus.

6. (a) (i) Factorise the cubic polynomial $z^3 + 8$

(α) field of real numbers

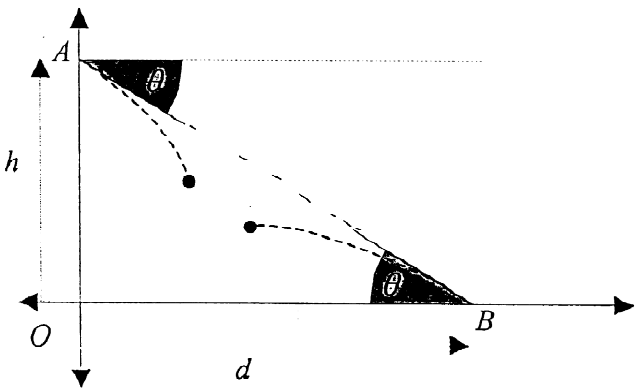
(β) over the field of complex numbers

Let W be one of the non-real complex roots of the equation $z^3 + 8 = 0$.

(ii) Show that $w^2 = 2w - 4$

(iii) Hence simplify $(2w - 4)^6$

(b) The diagram shows the point A at a height h vertically above the point O. It also shows the point B which is positioned at a horizontal distance d from O. A projectile is fired from A directly at point B with a velocity V. At the same instant a projectile is fired from point B directly at point A with the same velocity V.



Let θ be the angle between the horizontal at A and the angle of projection.

(i) Show carefully that the equations of motion of the two projectiles are given by

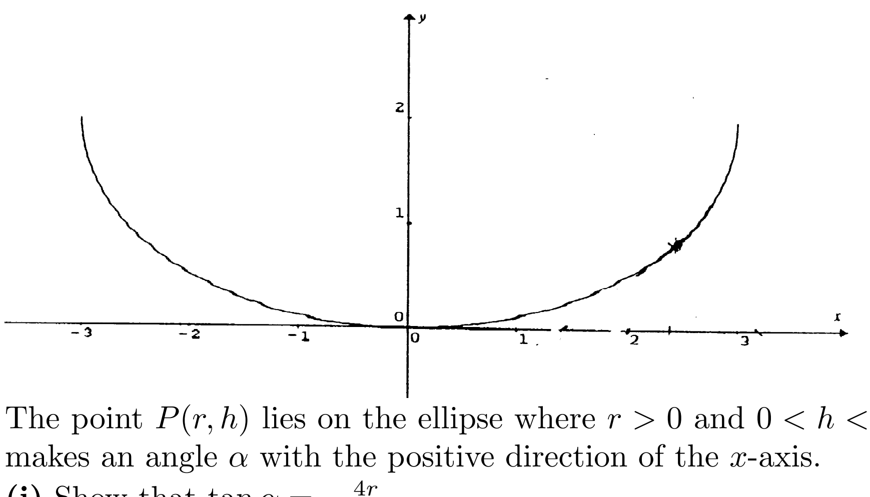
$$x_A = Vt\cos\theta \qquad\qquad x_B = d - Vt\cos\theta$$

$$y_A = h - Vt\sin\theta - \frac{gt^2}{2} \qquad\qquad y_B = Vt\sin\theta - \frac{gt^2}{2}$$

(ii) Show that the two particles will always meet.
(iii) Show that the height H at which they meet is given by $H = \frac{h}{2} - \frac{g(h^2+d^2)}{gV^2}$
(iv) Find the range of values of V such that the two projectiles meet above the x-axis.

7. **(a)** The equation $x^3 + 2x^2 + bx - 16 = 0$ has roots α, β and γ such that $\alpha\beta = 4$.
(i) Show that $b = -20$.
(ii) Find the cubic equation with roots given by α^2, β^2 and γ^2.
(iii) Hence find the value of $\alpha^3 + \beta^3 + \gamma^3$
(b) Consider the roots to the equation $z^n - 1 = 0$. These roots are plotted on an Argand diagram. The points represented by these roots are joined to form a regular n-sided polygon.
(i) Show that the area of this polygon is given by $A_n = \frac{n}{2}\sin\frac{2\pi}{n}$.
(ii) Show that the perimeter of the polygon is given by $P_n = 2n\sin\frac{\pi}{n}$.
(iii) Show that $P_n > 2A_n$ for all positive integers n.
(iv) Show that $\lim_{n\to\infty} A_n = \pi$ using the result that $\lim_{h\to 0}\frac{\sin h}{h} = 1$ and substituting an appropriate expression for n.
(v) Find the $\lim_{n\to\infty} P_n$.

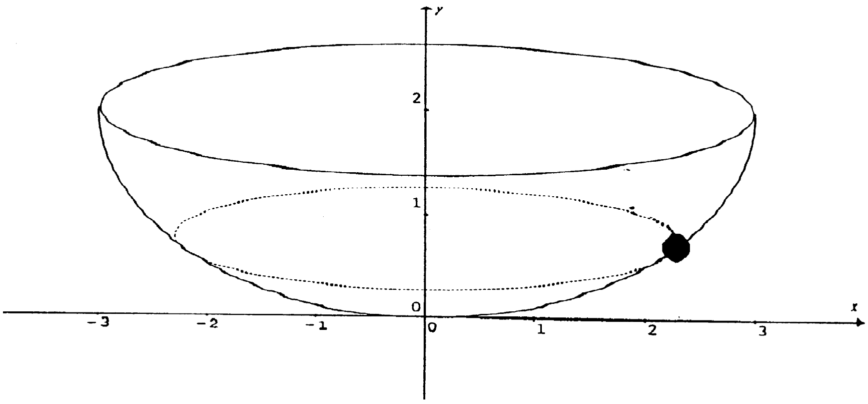
8. **(a)** The semi-ellipse given by $\frac{x^2}{9} + \frac{(y-2)^2}{4} = 1$ where $0 \leq y \leq 2$ is drawn below:



The point $P(r, h)$ lies on the ellipse where $r > 0$ and $0 < h < 2$. The tangent at P makes an angle α with the positive direction of the x-axis.
(i) Show that $\tan\alpha = \frac{4r}{9(2-h)}$
(ii) Hence, show that $\tan\alpha = \frac{2\sqrt{4-(2-h)^2}}{3(2-h)}$

(iii) Show that the acute angle between the normal at the point P and the vertical line $x = r$ is equal to the angle between the tangent at P and the positive direction of the x-axis.

(b) The semi-ellipse in part **(a)** is rotated about the y-axis to form a solid of revolution. A particle of unit mass slides smoothly in a horizontal circle on the inner surface of the solid so that it passes through the point P. The particle moves with a constant linear speed v.



(i) Draw a force diagram showing the forces acting on the particle at P.
(ii) By resolving these forces horizontally and vertically show that $v^2 = \frac{g(4-(2-h)^2)}{2-h}$.
(iii) If $h = 1$ then find the normal reaction of the surface of the solid on the particle.
(iv) If the normal reaction is equal to $\sqrt{2}$ times the weight of the particle then find the height h of the particle.