Student Number: ....

Moriah College

2003
TRIAL EXAMINATION

MATHEMATICS

Extension 2
Examiner: J. Taylor

Time Allowed - 3 hours
(plus 5 minutes reading time)

Directions to Candidates

- Start each question in a new booklet
- Attempt ALL questions
- Show all necessary working
- Standard integrals are printed on the last page
- Board-approved calculators may be used
- Additional Answer Booklets are available

Note there is a detachable sheet with the diagrams for Question 8. This should be submitted with your solutions to that question.

Marks

Question 1

a) Find $\int \frac{dx}{1+\sin x+\cos x}$ **3**

b) Find $\int \frac{x}{\sqrt{x^4-1}}dx$ **3**

c) Find $\int \frac{3u^2+2}{u(u^2+1)}du$ **4**

d) Evaluate **5**

$$\int_{-2}^{2} \frac{x^2}{\sqrt{4-x^2}}dx$$

Question 2

a) Find $\int x\sec x\tan x dx$ **2**

b) The equation $x^3-x^2+x+3=0$ has roots α, β and γ.

i) Find the equation with roots $\frac{1}{\alpha},\frac{1}{\beta},\frac{1}{\gamma}$. **2**

ii) Find α, β, γ given that one root is $x=1+i\sqrt{2}$. **2**

c) You are given the curve $x^2+xy+y^2=12$

i) By solving for x, show that $|y| \le 4$ **2**

ii) Find the coordinates of the highest and lowest points. **2**

iii) Show the points in (ii) are stationary points. **3**

iv) By carefully examining its symmetry, sketch the curve. **2**

Question 3

a) A cylindrical container of radius r and height L is partially filled with a liquid whose volume is V. (Figure 1) If the container is rotated about its axis of symmetry with constant angular speed ω, then the surface of the liquid will be convex, as indicated in Figure 2. The closer the water is to the edge, the further it will rise up the container, causing the water at the centre to be lower.

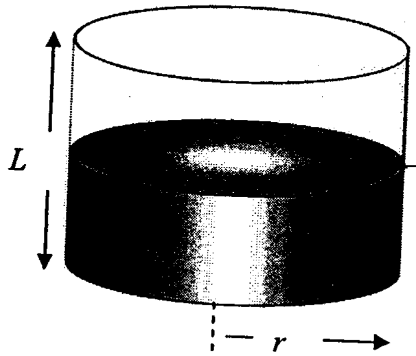


Figure 1

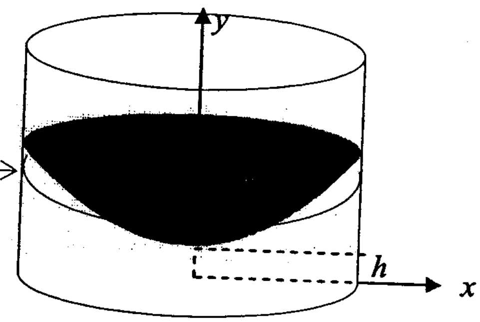


Figure 2

It can be shown that the surface of the liquid is a paraboloid of revolution generated by rotating the parabola

$$y = h + \frac{\omega^2 x^2}{2g}$$

about the y axis, where g is the acceleration due to gravity. Leave answers in terms of g.

(Note: you do not have to prove this, and the formulae for circular motion are not needed in this question).

i) By using the method of cylindrical shells, show that the volume of water under the parabola is **4**

$$V = \pi\left(r^2 h + \frac{\omega^2 r^4}{4g}\right)$$

ii) In terms of V and r, at what angular speed will the surface of the liquid touch the bottom? **2**

iii) The radius of a container is 2m, and the height of the container is 7m. The surface of the water, at the centre, is 5m below the top of the tank. At the edge of the container, the water is 4m below the top.

(1) Show that the volume of water is $10\pi\text{m}^3$. **2**

(2) Find the maximum angular speed of the container so that no water spills over the top. **3**

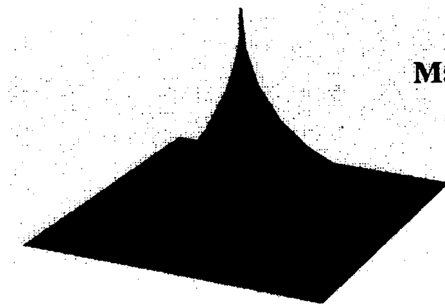
b) You are given the complex number $z = \dfrac{c+\sqrt{3}i}{c-\sqrt{3}i}$ where c is real. **4**

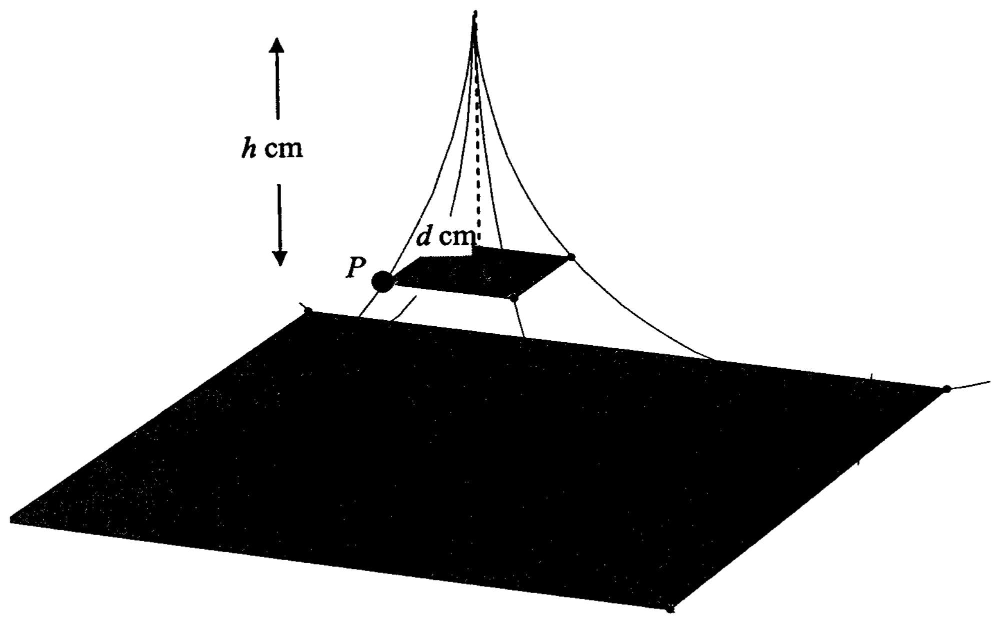
Find $|z|$ and hence describe the exact locus of z, if c varies from -1 to 1.

Marks

Question 4

a) A certain paperweight, which looks like a sharp Eiffel Tower, is drawn on the right and below. It is 10 cm high. The curved lines are all quarter circles, and P is a point on one of the curved lines. It has a square base.





A cross-section is drawn through P, parallel to the base. Let P be at a distance h cm below the peak.

i) Show that the distance d of a point P from the vertical axis is given by

$$d = 10 - \sqrt{100 - h^2}$$ **3**

ii) Hence find the volume of the paperweight. **8**

b) Sketch the graph of a function f such that **4**

$$f'(x) < 0 \text{ for all } x,$$

$$f''x) > 0 \text{ for } |x| > 1,$$

$$f''x) < 0 \text{ for } |x| < 1,$$

$$\text{and } \lim_{x \to \pm\infty} [f(x) + x] = 0$$

Question 5 **Marks**

a) If a, b are unequal positive numbers, prove that $a^a b^b > a^b b^a$ **2**

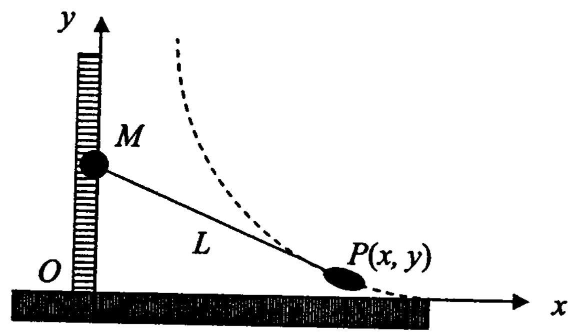
b) If x, y are positive and $x + y = 1$

i) prove that $xy \leq \dfrac{1}{4}$ **2**

ii) deduce that **3**

$$\left(x+\frac{1}{x}\right)^2+\left(y+\frac{1}{y}\right)^2 \geq \frac{25}{2}$$

c) A man M initially at the point O walks along a pier Oy pulling a row boat by a rope of length L. The boat is initially on the x axis L m from O. The man keeps the rope straight and taut. The path followed by the boat is such that the rope is always tangent to the curve (see the figure).



i) Show that if the path followed by the boat is the graph of the function $y = f(x)$, then **2**

$$f'(x) = \frac{dy}{dx} = \frac{-\sqrt{L^2 - x^2}}{x}$$

ii) Prove that $\dfrac{d}{d\theta}\left(\log\left|\sec\theta + \tan\theta\right|\right) = \sec\theta$ **2**

iii) Using the substitution $x = L\cos\theta$, or otherwise, determine the function $y = f(x)$. **4**

Question 6 **Marks**

a) A particle of mass m is released so that it falls vertically, under the influence of gravity, in a medium whose resistance is mkv^2, where v is the velocity and k is a constant.

i) Show that if the velocity is v m/s after x metres, then $\frac{dx}{dv} = \frac{v}{g - kv^2}$ **2**

ii) By evaluating an integral, show that the particle has a limiting velocity given by $V = \sqrt{\frac{g}{k}}$. **4**

b) This same particle is projected from ground level vertically upwards in the same medium with a velocity U.

4

i) Show that the maximum height reached is

$$H = \frac{1}{2k} \log \left[\frac{kU^2 + g}{g} \right]$$

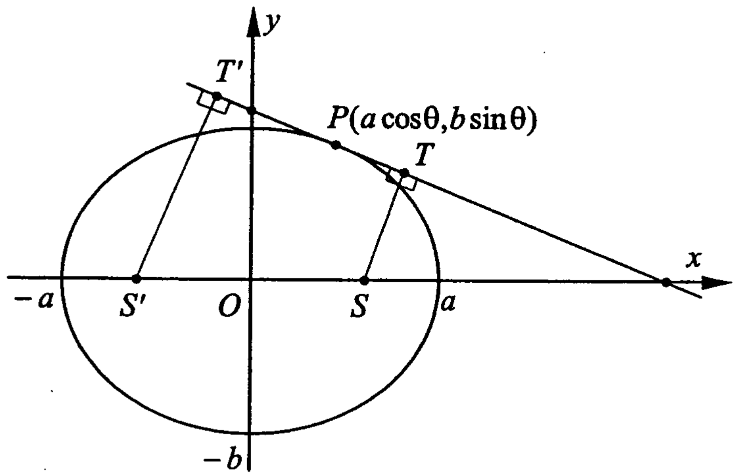
ii) Show that it returns to its starting point with a velocity **5**

$$\frac{UV}{\sqrt{U^2 + V^2}}$$

where V is the terminal velocity.

Question 7 **Marks**

a) The point $P(a\cos\theta, b\sin\theta)$ lies on the ellipse $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1$. The points T and T' are the feet of the perpendiculars from the foci S and S' respectively to this tangent.



i) Prove that the equation of the tangent at P is **3**

$$\frac{x\cos\theta}{a}+\frac{y\sin\theta}{b}=1$$

ii) Prove that $ST=\dfrac{|e\cos\theta-1|}{\sqrt{\dfrac{\cos^2\theta}{a^2}+\dfrac{\sin^2\theta}{b^2}}}$ where e is the eccentricity of the ellipse. **2**

iii) Show that $ST\times S'T'=b^2$. **4**

Question 7 continues over the page

Marks

b) A weight of mass m kg is placed at P, the middle of a string of length $2l$ metres. The string is fastened to two points A and B which are the same distance above a horizontal plane. A and B are so placed that the string makes an angle α with the horizontal (figure 1).

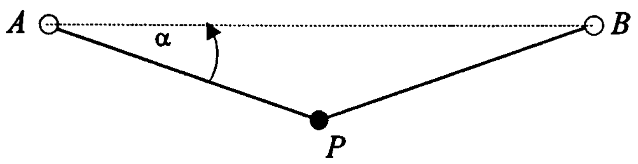


Figure 1

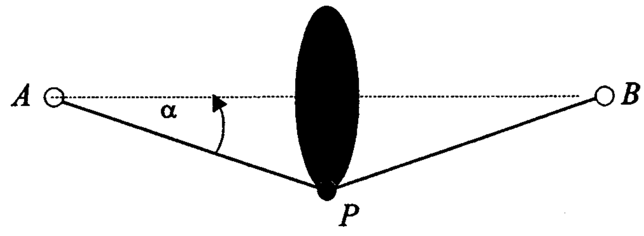
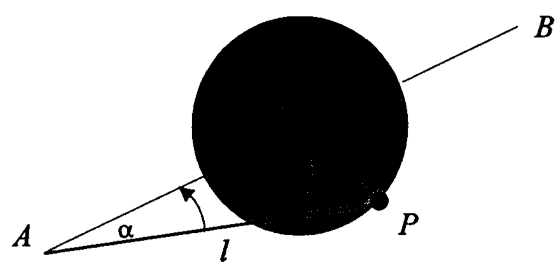


Figure 2

The mass is then twirled around the axis AB so that it traces out a circle with centre C, the midpoint of AB, and radius r. The mass moves with constant angular velocity $\omega = \dfrac{d\theta}{dt}$, where θ is the angle between the radius of the circle and the vertical (figure 2 and 3).



i) If T is the tension in the string, explain why $mr\omega^2 = 2T\sin\alpha - mg\cos\theta$ **2**

ii) By considering values of θ, deduce that the maximum tension in the string is **2**

$$\frac{m}{2}\left(l\omega^2 + \frac{g}{\sin\alpha}\right).$$

iii) Find the minimum angular velocity necessary so that the mass describes a complete circle. **2**

Note: The following diagrams are reproduced on the next page which can be removed and drawn upon. Insert this page into your Question 8 booklet.

Question 8 — **Marks**

a) i) **4**

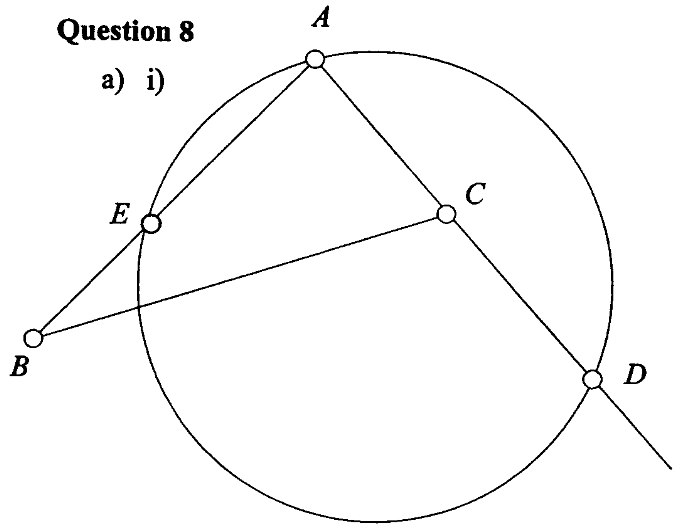


Figure 1

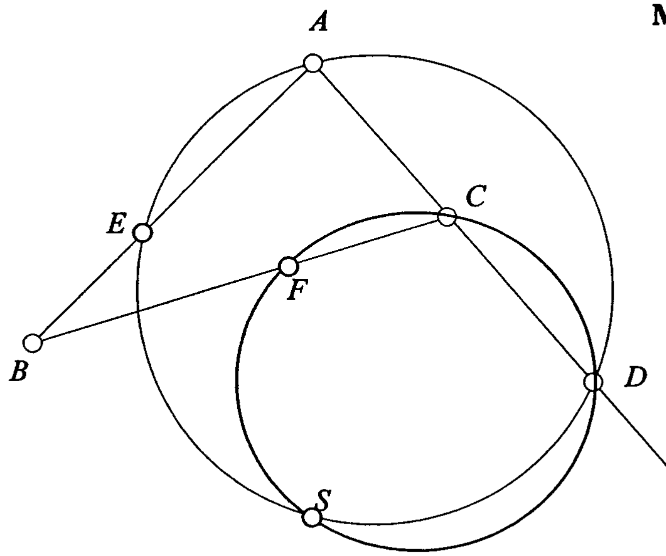
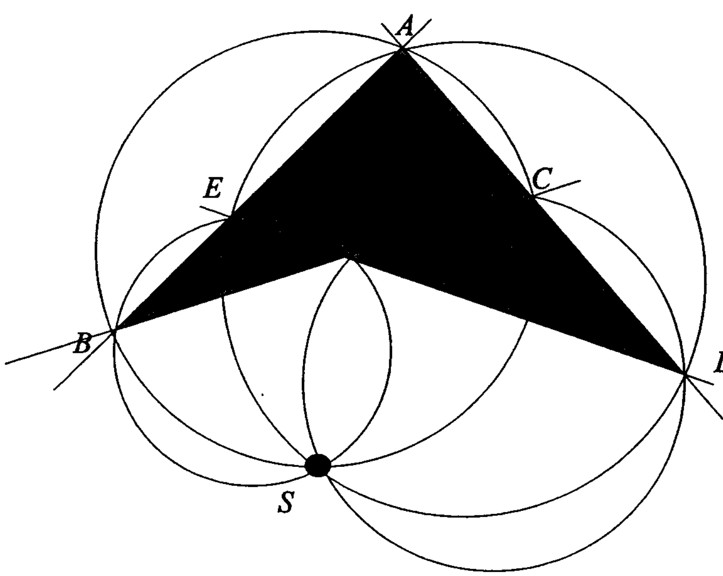


Figure 2

ABC is a triangle. D is a point on AC extended. A circle is drawn through A and D to cut AB at E (figure1). A second circle, with no particular radius, is now drawn to pass through D and C cutting BC at F and the first circle in S (figure 2).

Join ES and FS. Hence or otherwise prove that $BEFS$ is a cyclic quadrilateral.

ii) Four general lines are drawn, intersecting to form four triangles. The circumcircles of the triangles are then drawn, as in the figure below. **2**

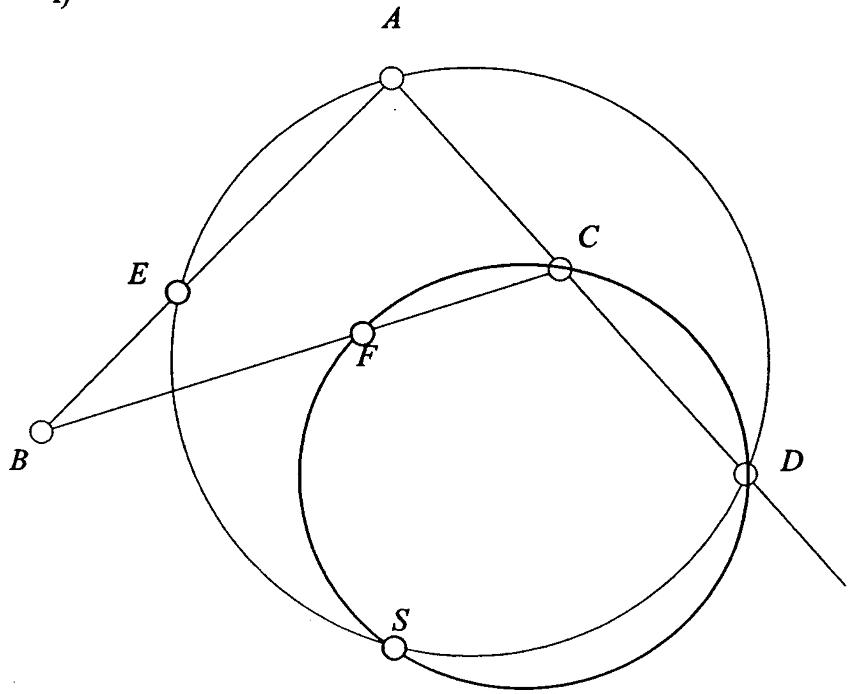


Prove these four circumcircles meet at a common point (S in the given diagram)

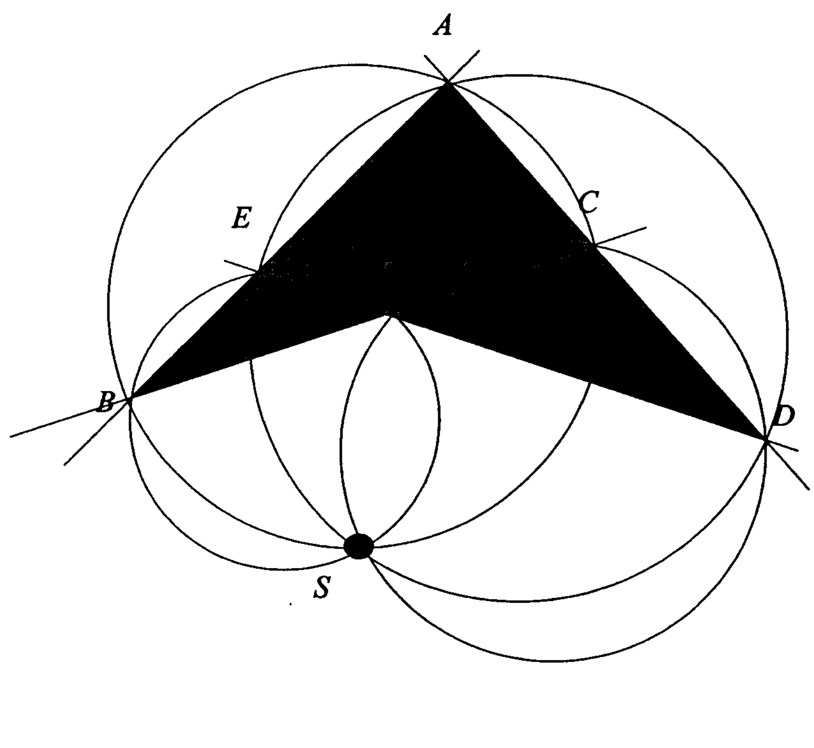
Question 8 continues over the page

Diagrams for Question 8 a) **Student Number ....**

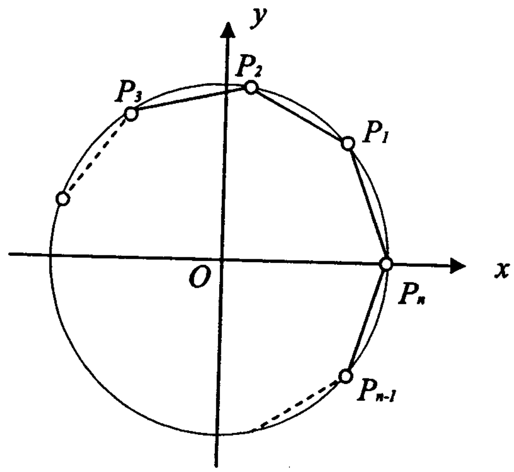
i)



ii)



b) i) Prove that the n nth roots of unity are integer powers of α, where $\arg(\alpha) = \frac{2\pi}{n}$. **Marks** 2



ii) $P_1, P_2, P_3, P_4, \ldots, P_n$ represent the complex numbers $z_1, z_2, z_3, z_4, \ldots, z_n$, and are the vertices of a regular polygon on a unit circle. Prove that 5

$$(z_1 - z_2)^2 + (z_2 - z_3)^2 + \ldots + (z_n - z_1)^2 = 0$$

iii) Deduce that 2

$$\sum_{r=1}^{n} z_r^{\ 2} = z_1 z_2 + z_2 z_3 + \ldots + z_{n-1} z_n + z_n z_1$$