NEWCASTLE GRAMMAR SCHOOL

Year 12

2004

Trial HSC Examination

Extension One Mathematics

Instructions

- Reading time – 5 minutes
- Working time – 2 hours
- Write using black or blue pen
- Board-approved calculators may be used
- A table of standard integrals is provided at the back of this paper
- All necessary working should be shown in every question
- Total marks- 84
- Attempt questions 1 to 7
- All questions are of equal value

Question One

(a) Solve $\frac{6}{x} \geq x-1$ and graph your solution on a number line. 3 marks

(b) A is the point (–2,3). B is the point (10,11). Find the ratio in which the point P(7,9) divides the interval AB. 3

(c) Evaluate $\int_0^{\frac{\pi}{3}} 3\sin x\cos^2 x\,dx$ using the substitution $t=\cos x$ 3

(d) Find, correct to the nearest minute, the size of the acute angle between the lines $y=2x-3$ and $x+3y-1=0$ 2

(e) Evaluate $\lim_{x\to\infty} \frac{2-3x-4x^2}{3x^2+5x}$ 1

Question Two (Start a new booklet)

(a) Let $f(x)=\sin^{-1}x+\cos^{-1}x$ 3

Find (i) $f'(x)$

(ii) $\int_0^1 f(x)dx$

(b) Differentiate $\tan^{-1}(\frac{1}{x})$ with respect to x. 2

(c) Use the principle of Mathematical Induction to prove that $5^n+2(11^n)$ is a multiple of 3 for all positive integer values of n. 4

(d) If α,β and γ are the roots of $x^3-3x+1=0$, find the values of 3

(i) $\alpha+\beta+\gamma$

(ii) $\alpha\beta\gamma$

(iii) $\frac{1}{\alpha}+\frac{1}{\beta}+\frac{1}{\gamma}$

Question Three (Start a new booklet)

(a) Prove that $\tan(\frac{\pi}{4}+x)=\frac{\cos x+\sin x}{\cos x-\sin x}$ 3

(b) A particle undergoes Simple Harmonic Motion about the origin O. Its displacement x cm from O at time t seconds is given by $x=3\cos(2t+\frac{\pi}{3})$. 5

(i) write expressions for the velocity and the acceleration of the particle in terms of t.

(ii) express the acceleration as a function of displacement.

(iii) write down the amplitude of the motion.

(iv) Find the maximum speed of the particle.

(c) (i) Prove that $\cos 2x=2\cos^2 x-1$ 3

(ii) Hence find $\int\cos^2 6x\,dx$

(d) Find the number of six-letter arrangements that can be made from the letters in the word **SYDNEY.** 1

Question Four (Start a new booklet)

(a) P(6p,3p^2) and Q(6q,3q^2) are two points on the parabola $x^2=12y$. The chord PQ passes through the fixed point (4,–3). 9

The tangents at P and Q meet at T.

(i) draw a diagram showing the above information.

(ii) show that the equation of PQ is $(p+q)x-2y-6pq=0$.

(iii) Use the fact that PQ passes through (4,–3) to show that 3(pq–1)=2(p+q).

(iv) Prove that the equation of the tangent at P is $y=px-3p^2$.

(v) Show the co-ordinates of point T are $x=3(p+q), y=3pq$.

(vi) Show that the locus of point T is the straight line $2x-3y+9=0$.

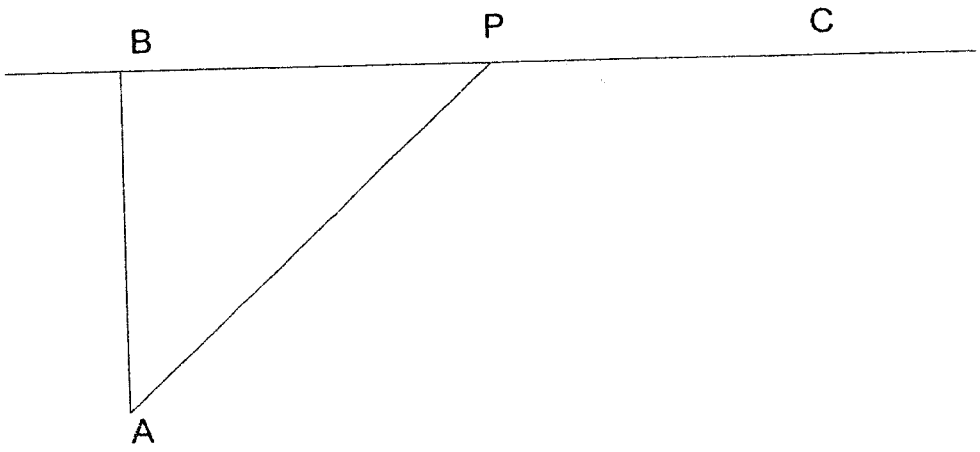
(b) The polynomial $P(x)=4x^3+ax^2+3x+b$ is divisible by $x-1$ and $x+2$. Find a and b. 3

Question Five (Start a new booklet)

(a) A class consists of 12 boys and 10 girls. How many ways are there of selecting a committee of 3 boys and 2 girls? 1

(b) Evaluate $\int_0^{\frac{2}{5}} \frac{dx}{\sqrt{16-25x^2}}$ 3

(c)



The diagram shows a straight road BC running due East. A four-wheel drive ambulance is in open country at A, 3 km due South of B. It must reach C, 9 km due East of B, as quickly as possible. The driver knows that she can travel at 80 km per hour in open country, and at 100 km per hour along the road. 6

She intends to proceed in a straight line to some point P on the road and then continue on the road to C. She wishes to choose P so that the total time for the journey is a minimum.

(i) If the distance BP is x km, show that the total journey time T hours from A to C via P in terms of x is given by

$$T = \frac{\sqrt{9+x^2}}{80} + \frac{9-x}{100}$$

(ii) Show that the minimum time for the total journey APC is $6\frac{3}{4}$ minutes.

(d) Given that $\sin x + \cos x = a$ and $\cos 2x = b$ prove that $a^4 - 2a^2 + b^2 = 0$. 2

Question Six(Start a new booklet)

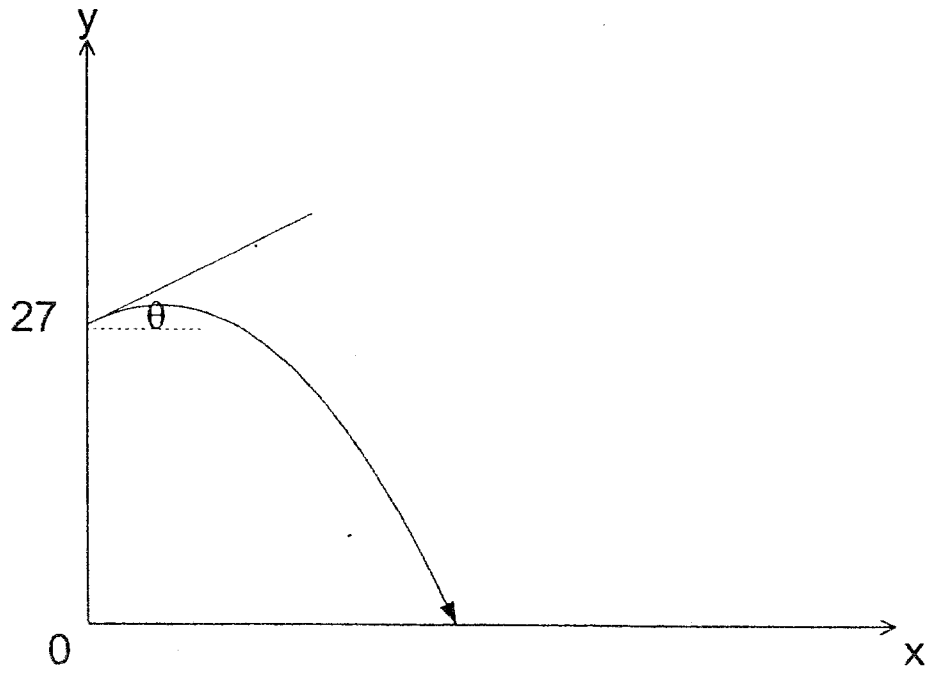
(a) A council meeting room contains a round table surrounded by 10 chairs. 3

These chairs are equally spaced around the room.

(i) A committee of 10 includes 3 people representing the Green party. Find the number of seating arrangements where all 3 can sit together.

(ii) Another committee of 10 elects a chairperson and minutes secretary and they sit around this table. Calculate the probability that the two people elected are sitting directly opposite each other. Explain your answer.

(b) 9



A stone is projected with velocity 10 metres per second at an angle of elevation $\theta = \tan^{-1}\frac{3}{4}$ from the top of a vertical cliff 27 metres high overlooking a lake.

(i) Commencing with the equations $\ddot{x} = 0$ and $\ddot{y} = -10$ and assuming the origin to be a point O at the base of the cliff, show that the expressions for the horizontal and vertical components of the stone's displacement from the origin after t seconds are given by $x = 8t, y = -5t^2 + 6t + 27$.

(ii) Calculate the time which elapses before the stone hits the lake, and find the horizontal distance of the point of contact from the base of the cliff.

(iii) What is the maximum height reached by the stone?

(iv) The path of the stone in the air is a parabolic arc. Find its equation in Cartesian form.

Question Seven(Start a new booklet)

(a) A spherical bubble is expanding so that its volume increases at a constant rate of 70 mm^3 per second. What is the rate of increase of its surface area when the radius is 10 mm? 3

(b) The acceleration of a particle in terms of its displacement x metres from a point O is given by $a = -e^{-2x}$. If at $x = 0$ the velocity of the particle is 1 metre per second find an expression for the velocity of the particle in terms of x. 3

(c) According to Newton's Law of Cooling, the rate at which a substance cools in moving air is proportional to the difference between the temperature of the substance and that of the air. This can be represented mathematically by the equation $\frac{dT}{dt} = k(T - A)$ where T is the temperature of the substance in degrees Celsius, t is the time elapsed in minutes, A is the air temperature and k is the cooling constant. 6

(i) Show that $T = A + Pe^{kt}$, where P is a constant, is a solution to the cooling equation $\frac{dT}{dt} = k(T - A)$.

(ii) If the temperature of the air is 25°C and the substance cools from 100°C to 50°C in 20 minutes, find the temperature after a further 20 minutes. (Answer to 1 decimal place.)

(iii) Find the length of time for the temperature to reach 40°C. (Answer to 1 decimal place).

Extension One Trial HSC 2004

Solutions.

1. (a) $\frac{6}{x} \geq x-1$

$x \neq 0$

$\frac{6}{x} = x-1$

$6 = x^2 - x$

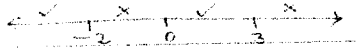
(3) $x^2 - x - 6 = 0$

$(x-3)(x+2) = 0$

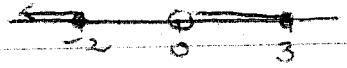
$x = 3, -2$

critical points are

$x = 0, 3, -2$



Solution is $x \leq -2, 0 < x \leq 3$



(c) $\int_0^{\frac{\pi}{3}} 3\sin x \cos^2 x \, dx$

$t = \cos x$

$dt = -\sin x \, dx$

$-3\,dt = 3\sin x \, dx$

$x = 0 \Rightarrow t = 1$

$x = \frac{\pi}{3} \Rightarrow t = \frac{1}{2}$

(3) $\int_0^{\frac{\pi}{3}} 3\sin x \cos^2 x \, dx$

$= \int_1^{\frac{1}{2}} t^2 \cdot -3 \, dt$

$= \int_1^{\frac{1}{2}} -3t^2 \, dt$

$= \left[-t^3\right]_1^{\frac{1}{2}}$

$= -\left(\frac{1}{2}\right)^3 - (-1^3)$

$= 1 - \frac{1}{8}$

$= -\frac{1}{8} - (-1)$

(b) $(x_1, y_1) = (-2, 3)$

$(x_2, y_2) = (10, 11)$

$(x, y) = (7, 9)$

(3) $x = \frac{mx_2 + nx_1}{m+n}$

$7 = \frac{10m - 2n}{m+n}$

$7m + 7n = 10m - 2n$

$9n = 3m$

$\frac{3}{1} = \frac{m}{n}$

ratio $= 3:1$

(d) $y = 2x - 3, \; m_1 = 2$

$x + 3y - 1 = 0, \; m_2 = -\frac{1}{3}$

$\tan\theta = \left|\frac{m_1 - m_2}{1 + m_1 m_2}\right|$

(2) $= \left|\frac{2 - -\frac{1}{3}}{1 + 2 \times -\frac{1}{3}}\right|$

$= \left|\frac{2\frac{1}{3}}{\frac{1}{3}}\right|$

$\tan\theta = 7$

$\theta = 81^\circ 52'$

1. (e) $\lim_{x \to \infty} \frac{2 - 3x - 4x^2}{3x^2 + 5x}$

(1) $= \lim_{x \to \infty} \frac{\frac{2}{x^2} - \frac{3}{x} - 4}{3 + \frac{5}{x}}$

$= -\frac{4}{3}$

2. (a) (i) $f(x) = \sin^{-1}x + \cos^{-1}x$

(1) $f'(x) = \frac{1}{\sqrt{1-x^2}} - \frac{1}{\sqrt{1-x^2}}$

$= 0.$

(ii) $f(x) = \sin^{-1}x + \cos^{-1}x$ — accept this or

$= \frac{\pi}{2}$

since $f'(x) = 0$

$f(x) = \int 0 \, dx$

$f(x) = c$

$\int_0^1 f(x)\,dx = \int_0^1 \frac{\pi}{2} \, dx$

$= \left[\frac{\pi}{2}x\right]_0^1$

(2) $= \frac{\pi}{2} - 0$

$= \frac{\pi}{2}.$

so $f(x) = \sin^{-1}x + \cos^{-1}x = c$

substitute a value of x, say $x = 0$

$\sin^{-1}0 + \cos^{-1}0 = c$

$c = \frac{\pi}{2}$

so $\int_0^1 f(x)\,dx = \int_0^1 \frac{\pi}{2} \, dx$

$= \left[\frac{\pi}{2}x\right]_0^1$

$= \frac{\pi}{2}.$

2(b) $y = \tan^{-1}\frac{1}{x}$

$$\frac{dy}{dx} = \frac{1}{1+\left(\frac{1}{x}\right)^2} \cdot -x^{-2}$$

(2)

$$= \frac{1}{1+\frac{1}{x^2}} \cdot -\frac{1}{x^2}$$

$$= \frac{x^2}{x^2+1} \cdot -\frac{1}{x^2}$$

$$\frac{dy}{dx} = -\frac{1}{x^2+1}$$

(c) Step 1 : Prove true for $n=1$

$n=1 \Rightarrow 5^n+2(11^n) = 5^1+2(11^1)$

$= 27$

which is divisible by 3

$\therefore$ result is true for $n=1$

Step 2 : Assume result is true for $n=k$

Prove true for $n=k+1$

(4 - 1) Assume $\frac{5^k+2(11^k)}{3} = M$, M an integer.

ie $5^k = 3M - 2(11^k)$ — ①

Prove $5^{k+1}+2(11^{k+1})$ divisible by 3

$5^{k+1}+2(11^{k+1}) = 5 \cdot 5^k + 2(11 \cdot 11^k)$

$= 5[3M-2(11^k)] + 22 \cdot 11^k$ (from ①)

$= 15M - 10 \cdot 11^k + 22 \cdot 11^k$

$= 15M + 12 \cdot 11^k$

which is divisible by 3.

$\therefore$ result is true for $n=k+1$

2.(c) Step 3 : Since result is true for $n=1$, it is true for $n=2$; since result is true for $n=2$, it is true for $n=3$; and so on for all positive integer values of n.

(3) (d) (i) $\alpha+\beta+\gamma = -\frac{b}{a}$

$= 0$

(ii) $\alpha\beta\gamma = -\frac{d}{a}$

$= -1$

(iii) $\frac{1}{\alpha}+\frac{1}{\beta}+\frac{1}{\gamma} = \frac{\beta\gamma+\alpha\gamma+\alpha\beta}{\alpha\beta\gamma}$

$= \frac{-3}{-1}$

$= 3$

3.(a) $\tan\left(\frac{\pi}{4}+x\right) = \frac{\cos x+\sin x}{\cos x-\sin x}$

$LHS = \frac{\tan\frac{\pi}{4}+\tan x}{1-\tan\frac{\pi}{4}\tan x}$

(3)

$= \frac{1+\frac{\sin x}{\cos x}}{1-\frac{\sin x}{\cos x}}$

multiply numerator and denominator by $\cos x$

$= \frac{\cos x+\sin x}{\cos x-\sin x}$

$= RHS$

3.(b) (i) $x = 3\cos(2t + \frac{\pi}{3})$

(2) $\dot{x} = -6\sin(2t + \frac{\pi}{3})$

$\ddot{x} = -12\cos(2t + \frac{\pi}{3})$

(1) (ii) $\ddot{x} = -4\left[3\cos(2t + \frac{\pi}{3})\right]$

$\ddot{x} = -4x$

(1) (iii) $a = 3$

(iv) $v^2 = n^2(a^2 - x^2)$

maximum speed at $x = 0$

(1) $v^2 = 2^2(3^2 - 0^2)$

$v^2 = 36$

$v = \pm 6$

max speed $= 6\ \text{cm s}^{-1}$.

(c) (i) $\cos 2x = \cos^2 x - \sin^2 x$

(1) $= \cos^2 x - (1 - \cos^2 x)$

$= 2\cos^2 x - 1$

(ii) $\cos 12x = 2\cos^2 6x - 1$

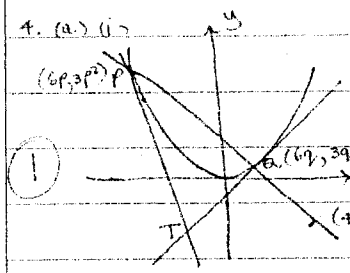
$\cos^2 6x = \frac{1}{2} + \frac{1}{2}\cos 12x$

(2) $\int \cos^2 6x\, dx = \int \frac{1}{2} + \frac{1}{2}\cos 12x\ dx$

$= \frac{1}{2}x + \frac{1}{24}\sin 12x + C$

(1) 3.(d) $\frac{6!}{2!} = 360$

4.(a)(i)



(1)

(ii) gradient $PQ = \frac{3p^2 - 3q^2}{6p - 6q}$

$= \frac{3(p+q)(p-q)}{6(p-q)}$

$= \frac{p+q}{2}$

(2) equation of PQ is

$y - 3p^2 = \frac{p+q}{2}(x - 6p)$

$2y - 6p^2 = (p+q)x - 6p^2 - 6pq$

$(p+q)x - 2y - 6pq = 0$

(iii) $x = 4, y = -3 \Rightarrow$

(1) $4(p+q) + 6 - 6pq = 0$

$2(p+q) = 3pq - 3$

$2(p+q) = 3(pq - 1)$

(2) $x = 6p \Rightarrow \frac{dy}{dx} = \frac{6p}{6}$

$= p$.

equation tangent at P

$y - 3p^2 = p(x - 6p)$

$y - 3p^2 = px - 6p^2$

$y = px - 3p^2$

(v) $y = px - 3p^2$ — (1)

$y = qx - 3q^2$ — (2)

$px - 3p^2 = qx - 3q^2$

(2) $x(p-q) = 3p^2 - 3q^2$

$x = \frac{3(p-q)(p+q)}{p-q}$

$x = 3(p+q)$

into (1) $\Rightarrow$

$y = p[3(p+q) - 3p^2]$

$y = 3p^2 + 3pq - 3p^2$

$y = 3pq$

Coords of T:

$x = 3(p+q)$, $y = 3pq$

(vi) $x = 3(p+q)$ $\quad y = 3pq$

$p+q = \frac{x}{3}$ $\quad pq = \frac{y}{3}$

from (iii) $3(pq - 1) = 2(p+q)$

(1) $3(\frac{y}{3} - 1) = 2(\frac{x}{3})$

$y - 3 = \frac{2x}{3}$

$3y - 9 = 2x$

$2x - 3y + 9 = 0$

4(b) $f(x) = 4x^3 + ax^2 + 3x + b$

$f(1) = 0 \Rightarrow \quad 0 = 4 + a + 3 + b$

$a + b = -7 \quad -(1)$

$f'(-2) = 0 \Rightarrow \quad -32 + 4a - b + b = 0$

$4a + b = 38 \quad -(2)$ (3)

$3a = 45$

$a = 15$

$b = -22$

5.(a) $^{12}C_8 \times {}^{10}C_2 = 2\cancel{6}400$ (1)
$\quad\quad 9900$

(b) $\int_0^{\frac{3}{5}} \frac{dx}{\sqrt{16 - 25x^2}}$

$= \int_0^{\frac{3}{5}} \frac{dx}{\sqrt{25\left(\frac{16}{25} - x^2\right)}}$

$= \frac{1}{5}\int_0^{\frac{3}{5}} \frac{dx}{\sqrt{\frac{16}{25} - x^2}}$ (3)

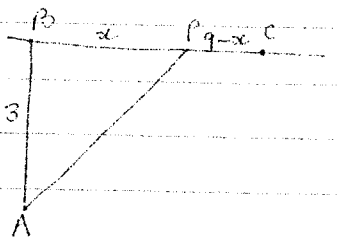
$= \frac{1}{5}\left[\sin^{-1}\frac{5x}{4}\right]_0^{\frac{3}{5}}$

$= \frac{1}{5}\left[\sin^{-1}\frac{1}{2} - \sin^{-1}0\right]$

$= \frac{1}{5} \times \frac{\pi}{2}$

$= \frac{\pi}{10}$.

5.(c) (i)



Time $= \frac{\text{Distance}}{\text{Speed}}$

$AP^2 = x^2 + 3^2$

$AP = \sqrt{9 + x^2}$ (2)

Time to go from A to P $= \frac{\sqrt{9+x^2}}{80}$

Time to go from P to C $= \frac{9-x}{100}$

Total time $= \frac{\sqrt{9+x^2}}{80} + \frac{9-x}{100}$

(ii) $T = \frac{(9+x^2)^{\frac{1}{2}}}{80} + \frac{9-x}{100}$

$\frac{dT}{dx} = \frac{1}{160}(9+x^2)^{-\frac{1}{2}} \cdot 2x \quad - \frac{1}{100}$

$= \frac{2x}{160\sqrt{9+x^2}} - \frac{1}{100}$

$\frac{dT}{dx} = 0 \Rightarrow \quad \frac{x}{80\sqrt{9+x^2}} - \frac{1}{100} = 0$

$\frac{x}{4\sqrt{9+x^2}} = \frac{1}{5}$

$5x = 4\sqrt{9+x^2}$

$25x^2 = 16(9+x^2)$

$25x^2 = 144 + 16x^2$

$9x^2 = 144$

$x^2 = \frac{144}{9}$

$x = \frac{12}{3}$

$x = 4$.

x	4^-	4	4^+
$\frac{dT}{dx}$	<0	0	>0

(4)

min value of T when $x = 4$

min $T = \frac{(9+4^2)^{\frac{1}{2}}}{80} + \frac{9-4}{100}$

$= \frac{5}{80} + \frac{1}{20}$

$= \frac{9}{80}$ hours

$= \frac{9}{80} \times 60$ min

Time $= 6\frac{3}{4}$ minutes.

5.(d) $a^4 - 2a^2 + b^2 = a^2(a^2 - 2) + b^2$

$= (\sin x + \cos x)^2\left[(\sin x + \cos x)^2 - 2\right] + (\cos 2x)^2$

$= (\sin^2 x + 2\sin x\cos x + \cos^2 x)(\sin^2 x + 2\sin x\cos x + \cos^2 x - 2) + \cos^2 2x$ (2)

$= (\sin 2x + 1)(\sin 2x - 1) + \cos^2 2x$

$= \sin^2 2x - 1 + \cos^2 2x$

$= 1 - 1$

$= 0$.

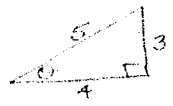
6. (a) (i) $7! \times 3!$

(2) $= 30240$

(ii) The probability that the minutes secretary is opposite the chairperson is $\frac{1}{9}$.

(1)

(b) (i)



at $t=0$, $\dot{x} = V\cos\theta$
$= 10 \times \frac{4}{5}$
$= 8$

$\dot{y} = V\sin\theta$
$= 10 \times \frac{3}{5}$
$= 6$

(3)

$\ddot{x} = 0$
$\dot{x} = \int 0\, dt$
$\dot{x} = c_1$
$t=0, \dot{x} = 8 \Rightarrow c_1 = 8$
$\dot{x} = 8$
$x = \int 8\, dt$
$x = 8t + c_2$
$t=0, x=0 \Rightarrow c_2 = 0$
$x = 8t$

$\ddot{y} = -10$
$\dot{y} = \int -10\, dt$
$\dot{y} = -10t + c_3$
$t=0, \dot{y} = 6 \Rightarrow c_3 = 6$
$\dot{y} = -10t + 6$
$y = \int -10t + 6\, dt$
$y = -5t^2 + 6t + c_4$
$t=0, y = 27$
$\Rightarrow c_4 = 27$
$y = -5t^2 + 6t + 27$

(ii) $y = 0 \Rightarrow -5t^2 + 6t + 27 = 0$
$5t^2 - 6t - 27 = 0$
$(5t+9)(t-3) = 0$
$t = 3$ sec

(2)

$t = 3 \Rightarrow x = 8 \times 3$
$= 24$ m.

(iii)
$\dot{y} = 0 \Rightarrow -10t + 6 = 0$
$t = \frac{3}{5}$ sec

(2)

max height
$= -5 \times \left(\frac{3}{5}\right)^2 + 6 \times \frac{3}{5} + 27$
$= -5 \times \frac{9}{25} + \frac{18}{5} + 27$
$= 28\frac{4}{5}$ metres.

(iv) $x = 8t$, $y = -5t^2 + 6t + 27$
$t = \frac{x}{8} \Rightarrow y = -5\left(\frac{x}{8}\right)^2 + 6\left(\frac{x}{8}\right) + 27$
$y = -\frac{5}{64}x^2 + \frac{3}{4}x + 27$

(2)

7. (a) $\frac{dv}{dt} = 70$

Find $\frac{dS}{dt}$ when $r = 10$

$V = \frac{4}{3}\pi r^3$

$\frac{dV}{dr} = 4\pi r^2$

$S = 4\pi r^2$

$\frac{dS}{dr} = 8\pi r$

$\frac{dS}{dt} = \frac{dv}{dt} \cdot \frac{dS}{dv}$

$= \frac{dv}{dt} \cdot \frac{dS}{dr} \cdot \frac{dr}{dv}$

(3)

$= 70 \times 8\pi r \times \frac{1}{4\pi r^2}$

$= \frac{140}{r}$

when $r = 10$,

$\frac{dS}{dt} = 14$ mm^2 sec^{-1}

(b) $a = -e^{-2x}$

$\frac{d}{dx}\left(\frac{1}{2}v^2\right) = -e^{-2x}$

$\frac{1}{2}v^2 = \int -e^{-2x}\, dx$

$\frac{1}{2}v^2 = \frac{1}{2}e^{-2x} + c$

(3)

$x = 0, v = 1 \Rightarrow c = 0$

$\frac{1}{2}v^2 = \frac{1}{2}e^{-2x}$

$v^2 = e^{-2x}$

$v = \pm\sqrt{e^{-2x}}$

$v = \pm e^{-x}$.

from initial conditions

$v = e^{-x}$.

7. (a) (i) $T = A + Pe^{kt}$

$\frac{dT}{dt} = kPe^{kt}$

(1) $k(T-A) = k(A+Pe^{kt}-A)$

$= kPe^{kt}$

$\therefore T = A+Pe^{kt}$ is a solution to $\frac{dT}{dt} = k(T-A)$

(ii) $A = 25 \Rightarrow T = 25 + Pe^{kt}$

$t=0, T=100 \Rightarrow 100 = 25 + P$

$P = 75$

$T = 25 + 75e^{kt}$

$t=20, T=50 \Rightarrow 50 = 25 + 75e^{kt}$

$25 = 75e^{kt}$

$\frac{25}{75} = e^{kt}$

(3) $\frac{1}{3} = e^{kt}$

$\log \frac{1}{3} = \log e^{kt}$

$\log \frac{1}{3} = 20k \log e$

$k = (\log \frac{1}{3}) \div 20$

$k = -0.054930614$

$T = 25 + 75 \cdot e^{-0.054930614t}$

$t=40 \Rightarrow T = 25 + 75e^{-0.054930614 \times 40}$

$T = 33.3^{\circ}$ C

(iii) $T = 25 + 75e^{-0.054930614t}$

$T=40 \Rightarrow 40 = 25 + 75e^{-0.054930614t}$

$\frac{15}{75} = e^{-0.054930614t}$

(2) $\log \frac{1}{5} = \log e^{-0.054930614t}$

$\log \frac{1}{5} = -0.054930614t$

$t = \frac{\log \frac{1}{5}}{-0.054930614}$

$t = 29.3$ minutes