Newcastle Grammar 2004 Maths Ext.2 Trial

**QUESTION 1** Use a SEPARATE Writing Booklet **Marks**

a) Evaluate $\int_0^1 te^{-t}\,dt$ 3

b) i) Find the real numbers $a$, $b$ and $c$ such that 2

$$\frac{1}{x(1+x^2)} \equiv \frac{a}{x} + \frac{bx+c}{1+x^2}$$

ii) Hence find $\int \frac{dx}{x(1+x^2)}$ 2

c) Evaluate $\int_0^4 \frac{x}{\sqrt{x+4}}\,dx$ 3

d) i) If $I_n = \int_0^{\frac{\pi}{2}} x^n \cos x\,dx$ show that, for $n > 1$ 3

$$I_n = \left(\tfrac{\pi}{2}\right)^n - n(n-1)I_{n-2}$$

ii) Hence find the area of the region bounded by the curve $y = x^4 \cos x$ and the $x$-axis for $0 \le x \le \frac{\pi}{2}$ 2



**QUESTION 2** Use a SEPARATE Writing Booklet **Marks**

a) The complex number $z$ moves such that $\operatorname{Im}\left(\frac{1}{z-i}\right) = 1$. 3

Show that the locus of $z$ is a circle and find its centre and radius.

b) i) Find the square roots of the complex number $5 - 12i$ 2

ii) Given that $z = \frac{1+\sqrt{5-12i}}{2+2i}$ and is purely imaginary, 2

find $z^{400}$

c) i) Shade the region on the Argand diagram containing all of the points representing the complex numbers $z$ such that 3

$$|z-1-i| \le 1 \quad \text{and} \quad \frac{-\pi}{4} \le \arg(z-i) \le \frac{\pi}{4}$$

ii) Let $w$ be the complex number of <u>minimum modulus</u> satisfying the inequalities of part i) above. 1

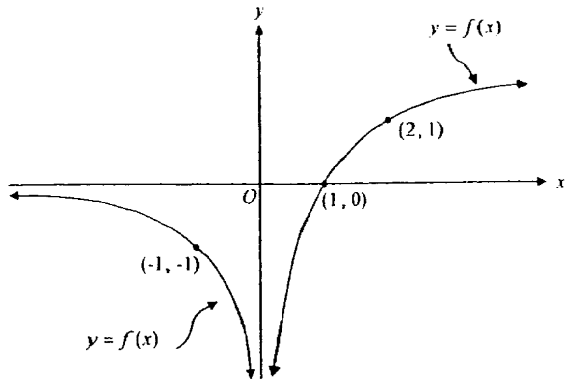
Express $w$ in the form $x + iy$.

d) Express $z = \frac{-1+i}{\sqrt{3}+i}$ in modulus/argument form and hence 4

evaluate $\cos\frac{7\pi}{12}$ in surd form.

**QUESTION 3** Use a SEPARATE Writing Booklet **Marks**

a) The diagram below shows the graph of the discontinuous function $y = f(x)$




Draw large (half page), separate sketches of the following

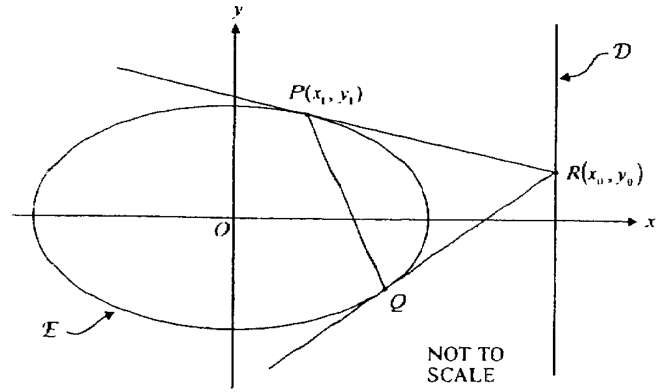
i) $y = -\sqrt{f(x)}$ **3**

ii) $y = \left|f(|x|)\right|$ **3**

iii) $y = \dfrac{1}{f(x)}$ **3**



b)




The ellipse $\mathcal{E}$ with equation $\dfrac{x^2}{25}+\dfrac{y^2}{16}=1$ has a directrix $\mathcal{D}$ as shown in the diagram. Point $R(x_0, y_0)$ lies on $\mathcal{D}$. $PQ$ is the chord of contact from $R$ where $P$ is the point $(x_1, y_1)$.

i) Write down the equation of $\mathcal{D}$ **1**

ii) Show that the equation of the tangent at $P$ is **3**

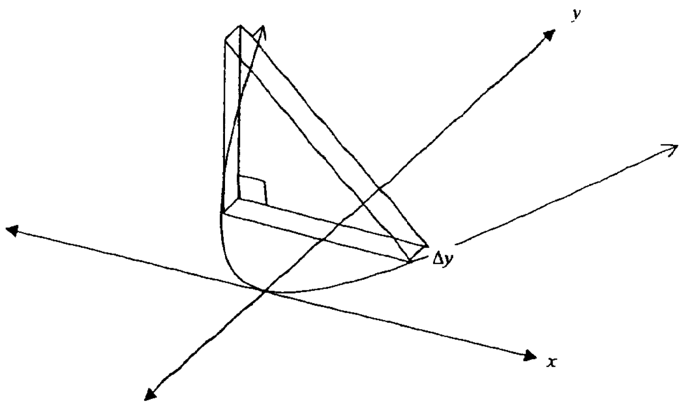
$$\frac{x_1 x}{25}+\frac{y_1 y}{16}=1$$

iii) The equation of $PQ$ is $\dfrac{x_0 x}{25}+\dfrac{y_0 y}{16}=1$ **2**

Show that the focus of $\mathcal{E}$ lies on $PQ$

**QUESTION 4** Use a SEPARATE Writing Booklet **Marks**

a) A solid is formed as shown below. Its base is in the $xy$-plane and is in the shape of the parabola $y = x^2$. The vertical cross-section is in the shape of a right angled isosceles triangle. **4**

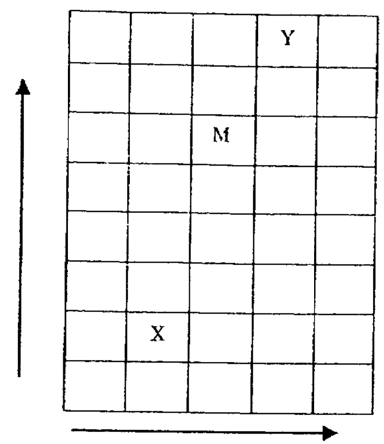



By using the method of slicing, calculate the volume of the solid between the values $y = 0$ and $y = 4$.

b) Find, using the method of cylindrical shells, the volume of the solid generated by rotating the region bounded by the curve $y = (x-2)^2$ and the line $y = x$ about the $x$-axis. **6**



c) On a special chess board, the squares are arranged in 8 rows and 5 columns as shown




A player can only move forwards or across in the directions shown by the arrows, one square at a time.

i) If a player is situated at X, in how many ways can the player reach the square labelled Y ? **3**

ii) In how many ways can a player move from X to Y if they <u>must</u> pass through M ? **2**

## QUESTION 5 — Use a SEPARATE Writing Booklet

**Marks**

a) The cubic equation $x^3 - x^2 + 4x - 2 = 0$ has roots $\alpha, \beta$ and $\gamma$

i) Find the equation with the roots $\alpha^2, \beta^2$ and $\gamma^2$ — **3**

ii) Find the value of $\alpha^2\beta^2 + \alpha^2\gamma^2 + \beta^2\gamma^2$ — **3**

b) If $P(x) = 4x^3 + 4x^2 + x + k$ for some real number $k$, find the values of $x$ for which $P'(x) = 0$. Hence find the values of $k$ for which the equation $P(x) = 0$ has more than one real root. — **4**

c) If $P(x) = 3x^4 - 11x^3 + 14x^2 - 11x + 3$ show that — **5**

$$P(x) = x^2\left\{3\left(x + \frac{1}{x}\right)^2 - 11\left(x + \frac{1}{x}\right) + 8\right\}$$

and hence solve $P(x) = 0$ over $C$ (complex numbers) and factorise $P(x)$ over $R$ (real numbers)



## QUESTION 6 — Use a SEPARATE Writing Booklet

**Marks**

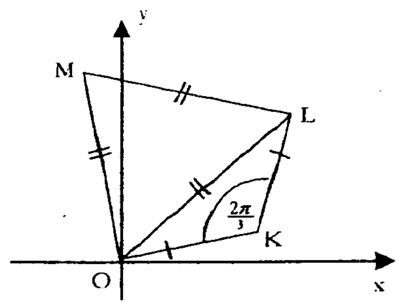
a) i) Show that the equation of the normal to the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ at the point $P(a\sec\theta, b\tan\theta)$ is — **4**

$$a\sin\theta\, x + by = (a^2 + b^2)\tan\theta$$

ii) The normal at the point $P(a\sec\theta, b\tan\theta)$ on the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ meets the $x$-axis at $G$. $PN$ is the perpendicular from $P$ to the $x$-axis. Prove that $OG = e^2 \times ON$, where $O$ is the origin. — **5**

b) The points K and M in a complex plane represent the complex numbers $\alpha$ and $\beta$ respectively. The triangle OKL is isosceles and $\angle OKL = \frac{2\pi}{3}$. The triangle OLM is equilateral. — **6**

Show that $3\alpha^2 + \beta^2 = 0$



## QUESTION 7 Use a SEPARATE Writing Booklet

**Marks**

a) Prove by induction that, for $n \geq 1$ **5**

$$\cos\frac{90^\circ}{2^n} = \frac{1}{2}\underbrace{\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{\ldots+\sqrt{2}}}}}}_{n-terms}$$

b) i) Prove that: **3**

$$\tan^{-1}(n+1) - \tan^{-1}(n) = \cot^{-1}\left(1+n+n^2\right)$$

ii) Hence, sum the series **3**

$$\cot^{-1}3 + \cot^{-1}7 + \cot^{-1}13 + \ldots + \cot^{-1}\left(1+n+n^2\right)$$

c) Using a graph, find the values of $x$ for which $f(x) > (f(x))^3$ where $f(x) = \frac{1}{2} + \sin x$ and $0 \leq x \leq 2\pi$ **4**



## QUESTION 8 Use a SEPARATE Writing Booklet

**Marks**

a) The tangent at $P(cp, \frac{c}{p})$ to the hyperbola $xy = c^2$ meets the lines $y = \pm x$ at $A$ and $B$ respectively. The normal at $P$ meets the axes at $C$ and $D$. If $M$ represents the area of $\Delta OAB$ and $N$ represents the area of $\Delta OCD$ show that $M^2N$ is a constant. **6**

b) i) Determine whether $f(x) = \frac{1-|x|}{|x|}$ is even, odd or neither. Justify your answer. **1**

ii) Sketch $y = f(x)$ **3**

iii) Hence, or otherwise, solve $f(x) \geq 1$ **3**

iv) Sketch $y = e^{f(x)}$ **2**