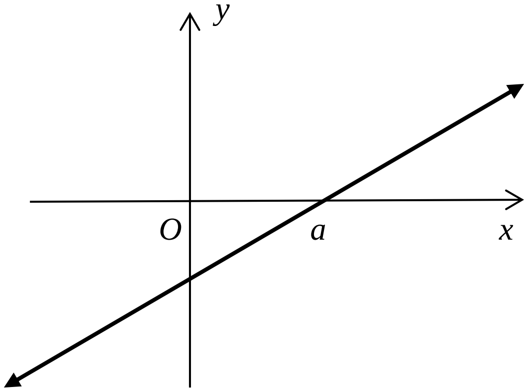
Section I

10 Marks
Attempt Question 1 – 10.
Allow approximately 15 minutes for this section.
Use the multiple-choice answer sheet for Question 1 – 10

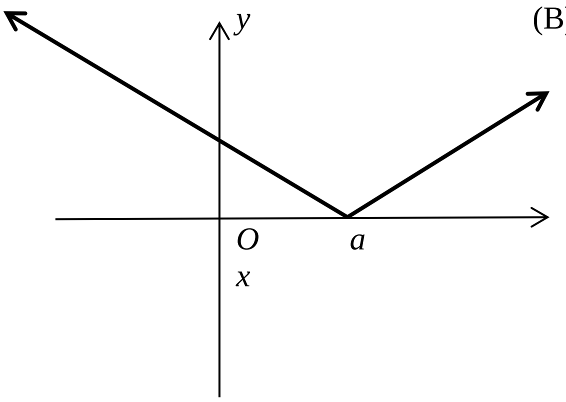
Question 1

The diagram illustrates the graph of $y = f(x)$.

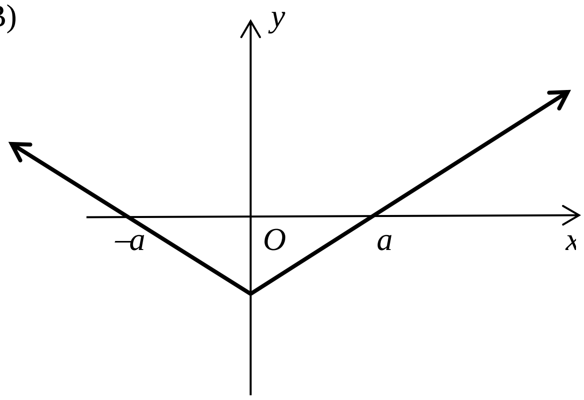


The graph of $y = f(|x|)$ is most likely to like:

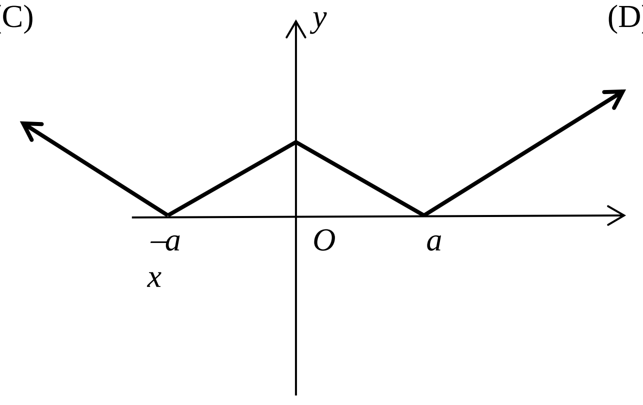
(A)



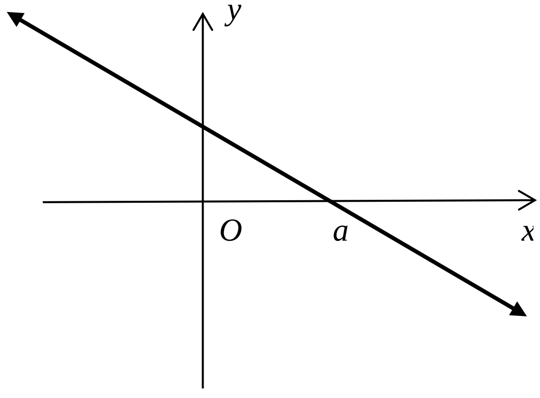
(B)



(C)



(D)



Question 2

The curve represented by the equation $\frac{x^2}{25} + \frac{y^2}{16} = 1$ is an ellipse with:

(A) eccentricity given by $e = \frac{3}{5}$ and foci at $(\pm 3, 0)$.

(B) eccentricity given by $e = \frac{5}{3}$ and foci at $(\pm 5, 0)$.

(C) eccentricity given by $e = \frac{3}{5}$ and foci at $(\pm 5, 0)$.

(D) eccentricity given by $e = \frac{4}{5}$ and foci at $(\pm 3, 0)$.

Question 3

Given that $w^3 = -1$ and that w is complex, the value of $\left(1 + w - w^2\right)^3$ is:

(A) -8
(B) 8
(C) 1
(D) -1

Question 4

The value of $\left[\frac{-1+\sqrt{-3}}{2}\right]^{29} + \left[\frac{-1-\sqrt{-3}}{2}\right]^{29}$ is:

(A) -1
(B) 1
(C) $-\sqrt{3}$
(D) -2

Question 5

What is the solution to the inequation: $\frac{x(5-x)}{x-4} \geq -3$?

(A) $2 \leq x < 4$ or $x \geq 6$.

(B) $1 \leq x < 4$ or $x \geq 5$.

(C) $4 < x \leq 6$ or $x \leq 2$.

(D) $4 > x \leq 5$ or $x \leq 1$.

Question 6

Consider the points $P\left(ct, \frac{c}{t}\right)$ and $Q\left(\frac{ct}{2}, \frac{2c}{t}\right)$ on the rectangular hyperbola $xy = c^2$. The locus of the midpoint, *M*, of *PQ* is:

(A) $xy = \frac{9}{8}c^2$

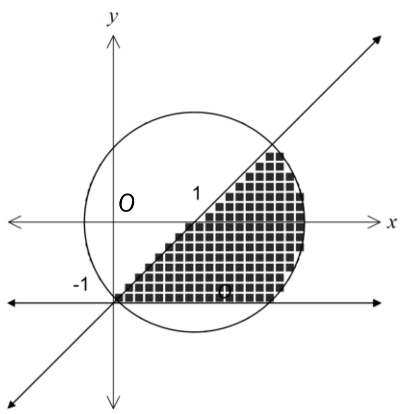
(B) $t = \frac{4x}{3c}$

(C) $xy = 8c^2$

(D) $xy = \frac{c^2}{2}$

Question 7

Consider the Argand diagram below.



Which inequality could define the shaded area?

(A) $|z-1| \le \sqrt{2}$ and $0 \le Arg(z-i) \le \frac{\pi}{4}$.

(B) $|z-1| \le \sqrt{2}$ and $0 \le Arg(z+i) \le \frac{\pi}{4}$.

(C) $|z-1| \le 1$ and $0 \le Arg(z-i) \le \frac{\pi}{4}$.

(D) $|z-1| \le 1$ and $0 \le Arg(z+i) \le \frac{\pi}{4}$.

Question 8

The gradient of the tangent to the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ at the point $P(a\sec\theta,\ b\tan\theta)$ is given by:

(A) $\frac{a\tan\theta}{b\sec\theta}$

(B) $-\frac{b\sec\theta}{a\tan\theta}$

(C) $-\frac{a\tan\theta}{b\sec\theta}$

(D) $\frac{b\sec\theta}{a\tan\theta}$

Question 9

Which of the following is an expression for $\int \frac{1}{\sqrt{x^2-6x+10}}\,dx$?

(A) $\ln\left(x-3-\sqrt{x^2-6x+10}\right)+c$

(B) $\ln\left(x+3-\sqrt{x^2-6x+10}\right)+c$

(C) $\ln\left(x-3+\sqrt{x^2-6x+10}\right)+c$

(D) $\ln\left(x+3+\sqrt{x^2-6x+10}\right)+c$

Question 10

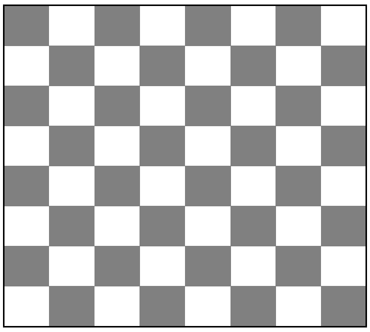
How many rectangles are in the chess board illustrated?

(A) 1024

(B) 2304

(C) 2025

(D) 1296



END OF SECTION I

Section II

Total Marks 90
Attempt Question 11 – 16.
Allow approximately 2 hours & 45 minutes for this section.

Answer all questions, starting each new question on a new booklet with your **student number** and the question number on the front cover.

All necessary working must be shown in each and every question.

Marks

Question 11. (15 marks) Use a SEPARATE writing booklet.

(a) Given $z = 2 + i$ and $w = -1 + 2i$, find:

(i) $|w|$ **1**

(ii) $\overline{z}$ **1**

(b) Illustrate the loci in the complex plane given by these equations:

(i) $|z| = |z - 6 - 3i|$ **2**

(ii) $\arg(z + 2) = \arg(z - 2 - 5i)$ **3**

(iii) $\text{Re}(z^2) = 4$ **2**

(c) Suppose z_1, z_2 and z_3 are three complex numbers, each of modulus 1, such that $z_1 + z_2 + z_3 = 0$. Suppose also that z is a complex number of modulus 3. Show that:

(i) $|z - z_1|^2 = 10 - (z\overline{z}_1 + \overline{z}z_1)$ **3**

(ii) $|z - z_1|^2 + |z - z_2|^2 + |z - z_3|^2 = 30$ **3**

Question 12. (15 marks) Use a SEPARATE writing booklet. **Marks**

(a) Solve the equation $x^4 + 2x^3 + x^2 - 1 = 0$, given that one root is **3**

$$-\frac{1}{2} + i\frac{\sqrt{3}}{2}.$$

(b) Show that if $f(x) = x^n - 1$ (for $n > 1$) then $f(x)$ has no multiple roots. **3**

(c) Given that $f(x) = x^2(x-1)$, draw a neat sketch showing any intercepts and asymptotes.

(i) $y = f(x)$ **1**

(ii) $y^2 = f(x)$ **2**

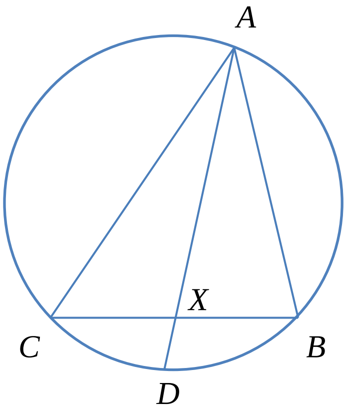
(iii) $y = \left[f(x)\right]^2$ **3**

(iv) $y = \dfrac{1}{f(x)}$ **3**

Question 13. (15 marks) Use a SEPARATE writing booklet.

Marks

(a)



ΔABC is inscribed a circle. The bisector of $\angle BAC$ meets BC at X and the circle at D.

Prove (i) ΔABX is similar to ΔADC **1**

(ii) $AB.AC = AD.AX$ **2**

(iii) $AB.AC = [AX]^2 + BX.XC$ **2**

(b)

(i) Evaluate $\int_0^{\frac{\pi}{2}} \cos^3 x \, dx$ **3**

(ii) Use the method of partial fractions to find $\int \frac{9x+1}{x^2-2x-3} dx$. **3**

(iii) If $I_n = \int x^n e^{ax} \, dx$,

(α) show that $I_n = \frac{1}{a} x^n e^{ax} - \frac{n}{a} I_{n-1}$ **2**

(β) Hence, or otherwise, evaluate $\int_1^3 x^2 e^{3x} \, dx$, giving your answer correct to 2 decimal places. **2**

Question 14. (15 marks) Use a SEPARATE writing booklet. **Marks**

(a) The region bounded by the graphs of $y = x^2$ and $y = x + 2$ is revolved about the line $x = 3$. Using the method of cylindrical shells, express the volume of the resulting solid as a definite integral. **(Note: Do not evaluate this integral).** **3**

(b) (i) Show that the cubic $f(x) = x^3 - 3x + 1$ has **exactly** one root, $x = \alpha$, between 0 and 1. **4**

(ii) Beginning with the approximation $x = 0$, use two applications of Newton's method to gain a better approximation to $x = \alpha$. Give your answer correct to three decimal places.

(c) Find the equation with rational coefficients whose roots are the reciprocals of the squares of the roots of $x^3 - x^2 - 14x + 24 = 0$ **3**

(d) Let $f(x) = \frac{1}{2}\left(1 - \sqrt{1-x}\right)$ for $0 \le x \le 1$ and define: **5**

$f_1(x) = f(x)$ and

$f_{n+1}(x) = f\left(f_n(x)\right)$, for $n = 1, 2, 3...$

Show by mathematical induction that, for any θ in the interval $0 \le \theta \le \frac{\pi}{2}$,

$f_n\left(\sin^2\theta\right) = \sin^2\left(\frac{\theta}{2^n}\right)$, for all positive integers *n*.

Question 15. (15 marks) Use a SEPARATE writing booklet. **Marks**

(a) (i) Prove that the equation of the tangent to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ at the point $P(x_1, y_1)$, is given by $\frac{xx_1}{a^2}+\frac{yy_1}{b^2}=1$. **2**

(ii) Hence, or otherwise, find the point of contact of the tangent $y = mx+\sqrt{a^2m^2+b^2}$ and the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$. **3**

(b) The foci of a hyperbola coincide with the foci of the ellipse $\frac{x^2}{25}+\frac{y^2}{9}=1$. **4**

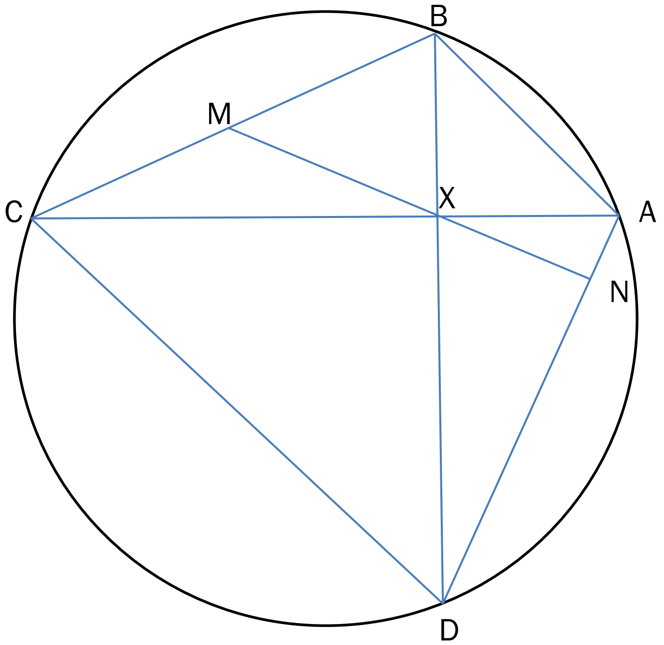
Find the equation of the hyperbola if its eccentricity is 2.

(c) (i) Prove that $\int_0^a F(x)\,dx = \int_0^a F(a-x)\,dx$. **1**

(ii) Hence, or otherwise, evaluate $\int_0^{\frac{\pi}{4}} \log_e\left(1+\tan\theta\right)\,d\theta$ **5**

Question 16. (15 marks) Use a SEPARATE writing booklet. **Marks**

(a)



ABCD is a cyclic quadrilateral. The diagonals *AC* and *BD* intersect at right angles at *X*. M is the midpoint of *BC*. *MX* produced meets *AD* at *N*.

(i) Copy the diagram showing the above information. **0**

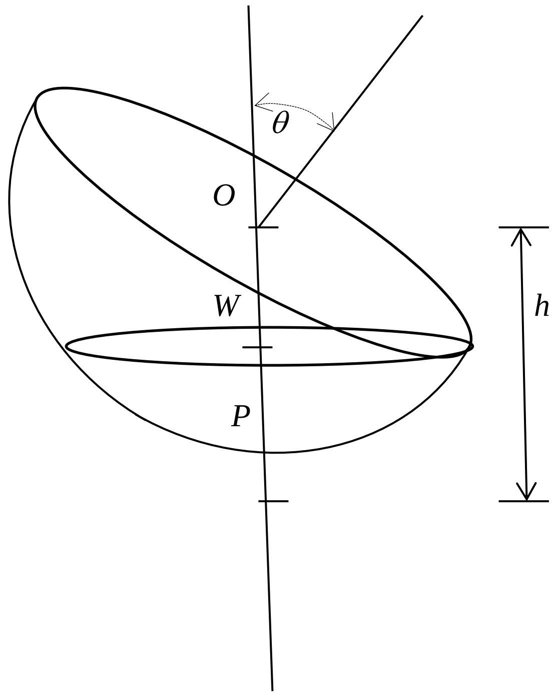
(ii) Show that $\angle MBX = \angle MXB$ **3**

(iii) Show that *MN* is perpendicular to *AD*. **4**

Question 16 continues on page 7

Question 16 (continued) **Marks**

(b)



The diagram shows a hemispherical bowl of radius *r*. The bowl has been tilted so that its axis is no longer vertical, but at an angle θ to the vertical. At this angle, it can hold a volume *V* of water.

The vertical line from the centre *O* meets the surface of the water at *W* and meets the bottom of the bowl at *B*. Let *P* be any point between *W* and *B* and let *h* be the distance *OP*.

(i) Explain why $V = \int_{r\sin\theta}^{r} \pi\left(r^2 - h^2\right)dh$. **2**

(ii) Hence show $V = \frac{r^3\pi}{3}\left(2 - 3\sin\theta + \sin^3\theta\right)$ **3**

(iii) The bowl has been tilted so that it is exactly half full. Using the result from Q 14 (b), find the angle θ correct to one tenth of a degree. **3**

END OF PAPER