Total marks – 120

Attempt All Questions

All questions are of equal value

<u>Answer each question in a SEPARATE writing booklet. Extra booklets are available.</u>

Question 1 *(15 Marks)* **Marks**

a) Find $\displaystyle\int \frac{1}{\sqrt{x^2+9}}\,dx$. 1

b) Use integration by parts to evaluate $\displaystyle\int_1^e \frac{\ln x}{\sqrt{x}}\,dx$ 3

c) Using the substitution $u = 1 - x$ evaluate $\displaystyle\int_0^{\frac{1}{2}} \frac{x}{(1-x)^2}\,dx$ 3

d) Find $\displaystyle\int \frac{dx}{x^2+4x+7}$. 2

e) (i) Show, using a suitable substitution that

$$\int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx.$$ 2

(ii) Hence evaluate $\displaystyle\int_0^{\frac{\pi}{4}} \frac{\cos x}{\cos x + \sin x}\,dx$. 4

Question 2 *(15 Marks)* **Use a SEPARATE writing booklet.** **Marks**

a) Let $z = \dfrac{7-i}{3-4i}$.

(i) Find $|z|$. **2**

(ii) Evaluate $\tan\left\{\tan^{-1}\left(\dfrac{4}{3}\right) - \tan^{-1}\left(\dfrac{1}{7}\right)\right\}$. **2**

(iii) Hence find the principal argument of $\dfrac{7-i}{3-4i}$ in terms of π. **2**

b) The point P represents the complex number z on the Argand diagram. Describe the locus of P when $\arg(z-2) = \arg(z+2) + \dfrac{\pi}{2}$. **2**

c) (i) Assuming the result $\cos 3\theta = 4\cos^3\theta - 3\cos\theta$, and using a suitable substitution, solve the equation $8x^3 - 6x + 1 = 0$. **3**

(ii) Hence find the value of

α) $\cos\dfrac{2\pi}{9} + \cos\dfrac{4\pi}{9} + \cos\dfrac{8\pi}{9}$. **1**

β) $\sec\dfrac{2\pi}{9} + \sec\dfrac{4\pi}{9} + \sec\dfrac{8\pi}{9}$. **3**

Question 3 *(15 Marks)* **Use a SEPARATE writing booklet.** **Marks**

a) Sketch the functions $g(x) = \sqrt{9-x^2}$ and $h(x) = x$ on the same axes. **3**

Use these graphs to sketch $y = f(x)$ where $f(x) = g(x).h(x)$. Hence sketch each of the following on separate number planes.

(i) $y = f(-x)$ **1**

(ii) $y = \dfrac{1}{f(x)}$ **2**

(iii) $y = |f(x)|$ **1**

(iv) $y^2 = f(x)$ **2**

b) (i) Show that $z = i$ is a root of the equation $(2-i)z^2 - (1+i)z + 1 = 0$. **1**

(ii) Find the other root of the equation in the form $z = a + ib$, where a and b are real numbers. **2**

c) Let p, q, r be the roots of the equation $x^3 - 4x + 7 = 0$. Write down the cubic equation in x whose roots are p^2, q^2 and r^2. **3**

Question 4 *(15 Marks)* **Use a SEPARATE writing booklet.** **Marks**

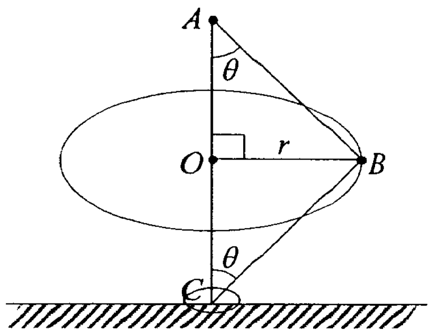
a) A particle of mass 1 kg is projected vertically upwards under gravity with a speed of 2c in a medium which the resistance to motion is $\frac{g}{c^2}$ times the square of the speed, where c is positive constant.

(i) Show that the maximum height (H) reached is 3

$$H = \frac{c^2}{2g}\ln 5.$$

(ii) Show that the speed with which the particle returns to its starting point is given by $v = \frac{2c}{\sqrt{5}}$. 4

b) Two light rigid rods AB and BC, each of length 0.5 m, are smoothly jointed at B and the rod is smoothly jointed at A to a fixed smooth vertical rod.



The joint at B has a particle of mass 2 kg attached. A small ring of mass 1 kg is smoothly joined to BC at C and can slide on the vertical rod below A. The ring rests on a smooth horizontal ledge at a distance $\frac{\sqrt{3}}{2}$ m below A. The system rotates about the vertical rod with constant angular velocity 6 radians per second. Find:

(i) the forces in the rod AB and BC; 5

(ii) the forces exerted by the ledge on the ring. (let $g = 10m/s^2$) 3

Question 5 *(15 Marks)* **Use a SEPARATE writing booklet.** **Marks**

a) i) Show that the tangent to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ at the point $P(a\cos\theta, b\sin\theta)$ has the equation $\frac{x\cos\theta}{a}+\frac{y\sin\theta}{b}=1$. 3

ii) This ellipse meets the y-axis at C and D. Tangents drawn at C and D on the ellipse meet the tangent in (i) at the points E, F respectively. Prove that $CE.DF = a^2$. 4

b) i) Show that if $y = mx + k$ is a tangent to the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$, then $m^2a^2 - b^2 = k^2$. 3

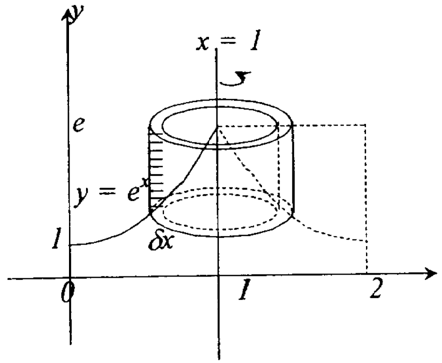
ii) Hence find the equation of the tangents from the point $(1, 3)$ to the hyperbola $\frac{x^2}{4}-\frac{y^2}{15}=1$ and the coordinates of their points of contact. 5

End of Question 5.

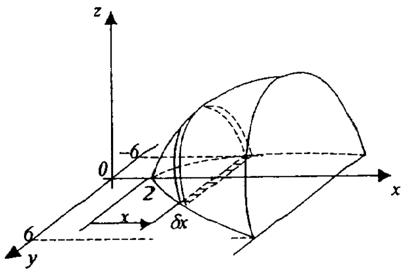
Please Turn Over.

Question 6 *(15 Marks)* **Use a SEPARATE writing booklet.** **Marks**

a) By taking strips parallel to the axis of rotation, use the method of cylindrical shells to find the volume of the solid obtained by rotating the region bounded by $y = e^x$, $y = e$ and the y-axis about the line $x = 1$. **6**



b) The base of a particular solid is the region bounded by the hyperbola $\frac{x^2}{4} - \frac{y^2}{12} = 1$ between its vertex $(2, 0)$ and the corresponding latus rectum. Every cross-section perpendicular to the major axis is a semicircle with diameter in the base of the solid.



i) Find the equation of the latus rectum. **2**

ii) Find the volume of the solid. **4**

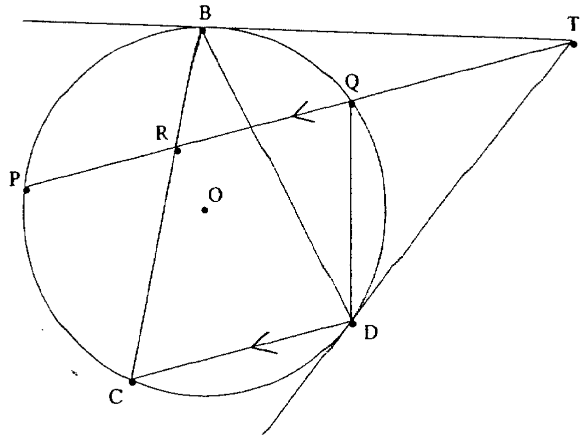
c) The points $P\left(cp, \frac{c}{p}\right)$ and $Q\left(cq, \frac{c}{q}\right)$ lie on the rectangular hyperbola $xy = c^2$. **3**

The chord PQ subtends a right angle at another point $R\left(cr, \frac{c}{r}\right)$ on the hyperbola.

Show that the normal at R is parallel to PQ.

Question 7 *(15 Marks)* **Use a SEPARATE writing booklet.** **Marks**

a)



PQ, CD are parallel chords of a circle, centre O. The tangent at D meets PQ extended at T. B is the point of contact of the other tangent from T. BC meets PQ at R.

(i) Copy the diagram.

(ii) Prove that $\angle BDT = \angle BRT$ and hence state why B, T, D and R are concyclic points. **3**

(iii) Prove $\angle BRT = \angle DRT$. **3**

(iv) Show that ΔRCD is isosceles. **2**

(v) Prove that $\Delta PRC \equiv \Delta QRD$. **3**

b) The equation $x^3 + 3px^2 + 3qx + r = 0$, where $p^2 \neq q$, has a double root. Show that **4**

$$4(p^2 - q)(q^2 - pr) = (pq - r)^2.$$

Question 8 *(15 Marks)* **Use a SEPARATE writing booklet.** **Marks**

a) A coin is tossed six times. What is the probability that there will be more tails on the first three of the six throws than on the last three throws? **3**

b) If m points are taken on a straight line and n points on a parallel line, how many triangles can be drawn each having its vertices at 3 of the given points? **3**

c) (i) Show that $\left(1-x^2\right)^{\frac{n-3}{2}}-\left(1-x^2\right)^{\frac{n-1}{2}}=x^2\left(1-x^2\right)^{\frac{n-3}{2}}$. **1**

(ii) Let $I_n=\int_0^1\left(1-x^2\right)^{\frac{n-1}{2}}\,dx$ where $n=0,1,2,\ldots,$ **3**

Show that $nI_n=(n-1)I_{n-2}$ for $n=2,3,4\ldots$

(iii) Let $J_n=nI_n.I_{n-1}$ for $n=1,2,3,\ldots$. **3**

By using mathematical induction, prove that

$J_n=\frac{\pi}{2}$ for $n=1,2,3,\ldots$

(iv) Briefly explain why $0<I_n<I_{n-1}$ for $n=1,2,3,\ldots$ **2**

END OF PAPER