Total Marks – 120
Attempt Questions 1-8
All questions are of equal value

Answer each question in a SEPARATE writing booklet. Extra writing booklets are available.

Question 1 *(15 marks)* Use a SEPARATE writing booklet. **Marks**

(a) Use Integration by parts to find $\int xe^{-3x}\,dx$ **2**

(b) Use the substitution $x = \frac{2}{3}\sin\alpha$ to prove that $\int_0^{\frac{2}{3}} \sqrt{4-9x^2}\,dx = \frac{\pi}{3}$ **3**

(c) Use the table of standard integrals to help evaluate $\int \frac{dx}{\sqrt{x^2-4x+29}}$ **2**

(d) Evaluate that $\int_4^6 \frac{2dt}{(t-1)(t-3)}$ **4**

(e) Use the substitution $t = \tan\frac{x}{2}$ to show that

$$\int_0^{\frac{\pi}{2}} \frac{dx}{5+4\cos x} = \frac{2}{3}\tan^{-1}\frac{1}{3}$$ **4**

Question 2 *(15 marks)* Use a SEPARATE writing booklet. **Marks**

(a) For the complex number $z = 1 - \sqrt{3}i$, express each of the following in the form $a + bi$, (a, b are real numbers). **3**

(i) $\bar{z}$

(ii) z^2

(iii) $\dfrac{1}{z}$

(b) If $z_1 = 1 - 2i$; $z_2 = 2 + i$ and $z = \dfrac{z_1}{z_2}$ find :- **3**

i) $|z|$

ii) $\arg(z)$

(c) Prove that $|z|^2 = z\bar{z}$ for all complex numbers z . **2**

(d) If ω is a complex cube root of unity,

(i) Write down the value of $1 + \omega + \omega^2$. **1**

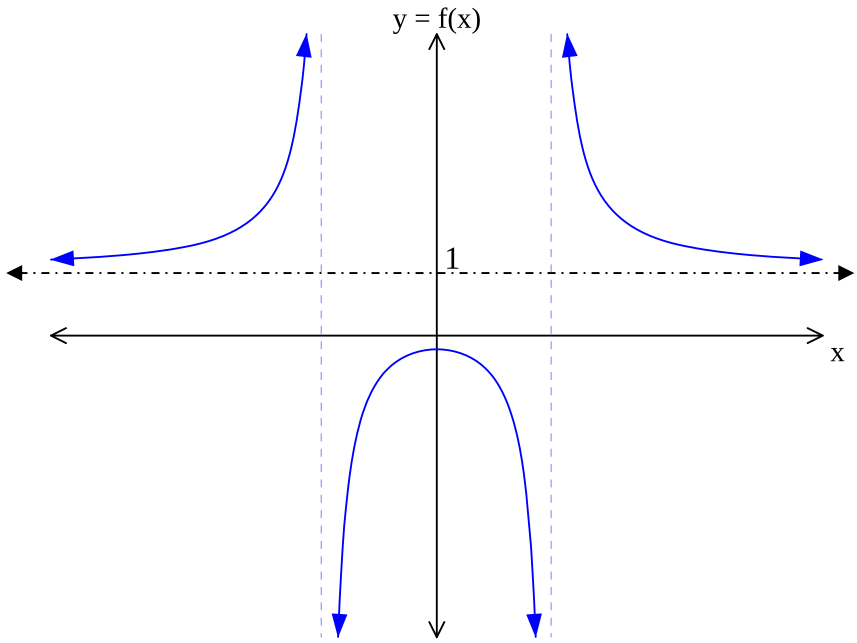
(ii) Simplify $\omega^4 + \omega^5 + \omega^6$. **1**

(e) Sketch on an Argand diagram the region in which z lies, showing all important features where $2 \le |z| \le 3$ and $\dfrac{\pi}{4} < \arg(z - i) < \dfrac{3\pi}{4}$. **2**

(f) P_1 and P_2 are points representing the complex numbers z_1 and z_2 on an Argand diagram. If OP_1P_2 is an isosceles triangle and angle P_1OP_2 is a right-angle, show that $z_1^2 + z_2^2 = 0$. **3**

Question 3 *(15 marks)* Use a SEPARATE writing booklet. **Marks**

(a) The diagram below shows the graph of the function $y = f(x)$ where $f(x) = \dfrac{1+x^2}{x^2-9}$.



Draw a separate sketch of each of the following graphs.
Use about one third of a page for each graph. Show all significant features.

(i) $y = [f(x)]^2$ **2**

(ii) $y = \dfrac{1}{f(x)}$ **2**

(iii) $y = f'(x)$ **2**

(iv) Draw $y = f(x)$ and $y = \sqrt{f(x)}$ on the same number plane. **2**

(v) $|y| = f(x)$ **2**

(b) For the curve defined by $3x^2 + y^2 - 2xy - 8x + 2 = 0$,

(i) Show that $\dfrac{dy}{dx} = \dfrac{3x - y - 4}{x - y}$. **2**

(ii) Find the coordinates of the points on the curve where the tangent to the curve is parallel to the line $y = 2x$. **3**

Question 4 *(15 marks)* Use a SEPARATE writing booklet. **Marks**

(a) The equation $x^3 + 2x^2 + bx - 16 = 0$ has roots α, β and γ such that $\alpha\beta = 4$.

(i) Show that $b = -20$. **2**

(ii) Find the equation with roots given by α^2, β^2 and γ^2 **2**

(b) Consider the polynomial $P(x) = (x-\alpha)^3 . Q(x)$, where $Q(x)$ is also a polynomial and α is a real zero of $P(x)$.

(i) Show that $P(\alpha) = P'(\alpha) = P''(\alpha) = 0$ **2**

(ii) Hence or otherwise, solve the equation
$8x^4 - 25x^3 + 27x^2 - 11x + 1 = 0$
given that it has a triple root. **2**

(c) (i) Show that the solutions of $z^6 + z^3 + 1 = 0$ are contained in the solutions of $z^9 - 1 = 0$. **2**

(ii) Sketch the nine solutions of $z^9 - 1 = 0$ on an Argand Diagram. (about one third of a page in size) **2**

(iii) Mark clearly on your diagram, the six roots $z_1, z_2, z_3, z_4, z_5, z_6$ of $z^6 + z^3 + 1 = 0$. **4**

(iv) Show that the sum of the six roots of $z^6 + z^3 + 1 = 0$ can be given by **2**
$2\left(\cos\frac{2\pi}{9} + \cos\frac{4\pi}{9} - \cos\frac{\pi}{9}\right)$

Question 5 *(15 marks)* Use a SEPARATE writing booklet. **Marks**

(a) For the curve with Cartesian equation $\frac{x^2}{9}+\frac{y^2}{4}=1$, **5**

(i) find the eccentricity and the coordinates of a focus
(ii) find the equation of the corresponding directrix
(iii) hence write down the coordinates of a focus and the equation of the corresponding directrix for the curve with the Cartesian equation $\frac{x^2}{4}+\frac{y^2}{9}=1$.

(b) Let $P(4\sec\theta, 3\tan\theta)$ be any point on the hyperbola $\frac{x^2}{16}-\frac{y^2}{9}=1$.

(i) Derive the equations of the tangent and normal at P and show that they are respectively: **4**

$3x\sec\theta-4y\tan\theta=12$ and $4x\tan\theta+3y\sec\theta=25\sec\theta\tan\theta$

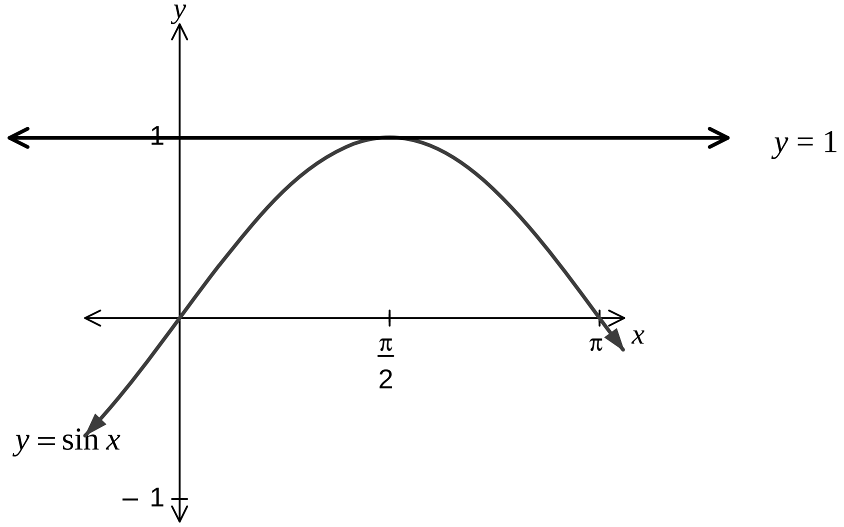
(ii) The tangent and normal at P meet the $y-axis$ at T and N respectively. Show that **2**

$T=(0,-3\cot\theta)$ and $N=(0,\frac{25}{3}\tan\theta)$.

(iii) Show that the circle with diameter NT passes through a focus. **4**

Question 6 *(15 marks)* Use a SEPARATE writing booklet. **Marks**

(a)



The area defined by $y \geq \sin x$, $0 \leq x \leq \frac{\pi}{2}$ and $0 \leq y \leq 1$ is rotated about the straight line $y = 1$.

(i) Copy the diagram above into your writing booklet and shade in the region defined by the simultaneous inequalities $y \geq \sin x$, $0 \leq x \leq \frac{\pi}{2}$ and $0 \leq y \leq 1$. **1**

(ii) Find the total volume of the solid formed, by taking slices perpendicular to the axis of rotation. **3**

(b) The horizontal base of a solid is an ellipse defined by the equation $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$. **5**

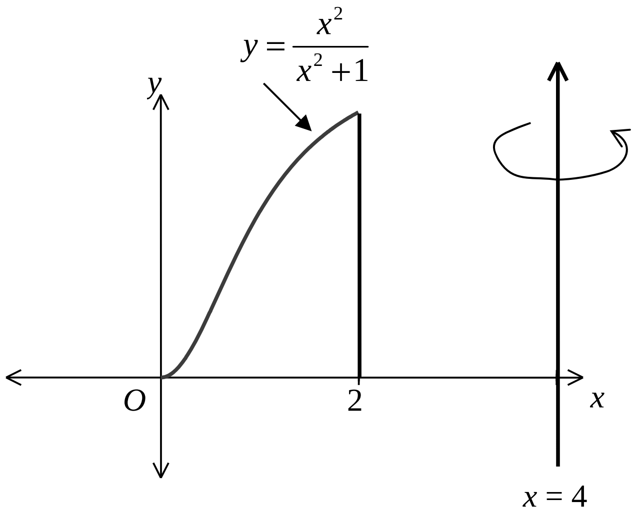
Vertical cross-sections taken perpendicular to the y axis are squares with one side in the horizontal base of the solid.

Find the volume of the solid formed in terms of a and b.

Question 6 continues on the next page …

Question 6 continued … **Marks**

(c) The region bounded by the curve $y = \dfrac{x^2}{x^2+1}$, the x axis and $0 \le x \le 2$, is rotated about the line $x = 4$ to form a solid.



(i) Using the method of cylindrical shells, explain why the volume δV of a typical shell distant x units from the origin and with thickness δx is given by **3**

$$\delta V = 2\pi\left(4-x\right)\left(1-\frac{1}{1+x^2}\right)\delta x .$$

(ii) Hence, find the total volume of the solid formed. **3**

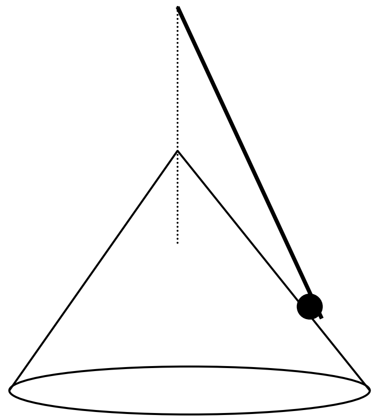
Question 7 *(15 marks)* Use a SEPARATE writing booklet. **Marks**

(a) A particle of mass m is set in motion with speed u. Subsequently the only force acting upon the particle directly opposes its motion and is of magnitude $mk(1 + v^2)$ where k is a constant and v is its speed at time t. **5**

(i) Show that the particle is brought to rest after a time $\frac{1}{k}\tan^{-1}u$.

(ii) Find an expression for the distance travelled by the particle in this time.

(b) A smooth circular cone, with its vertex up, and its axis vertical, has a semi-vertex angle of $60°$. A particle of mass 1 kg is attached by a light inelastic string from a point vertically above the vertex of the cone, and moves with constant speed $v\ m/s$ on the outer surface of the cone in a horizontal circle of radius 0.5 m. The string makes an angle of $30°$ with the vertical. Let the magnitude of the tension in the string be T newton, and let the magnitude of the reaction of the cone on the particle be R newton.



(i) Draw a diagram showing the forces acting on the particle, and the magnitude of the angles made by these forces with the vertical. **2**

(ii) By resolving forces in two directions write down equations of motion for the particle. **2**

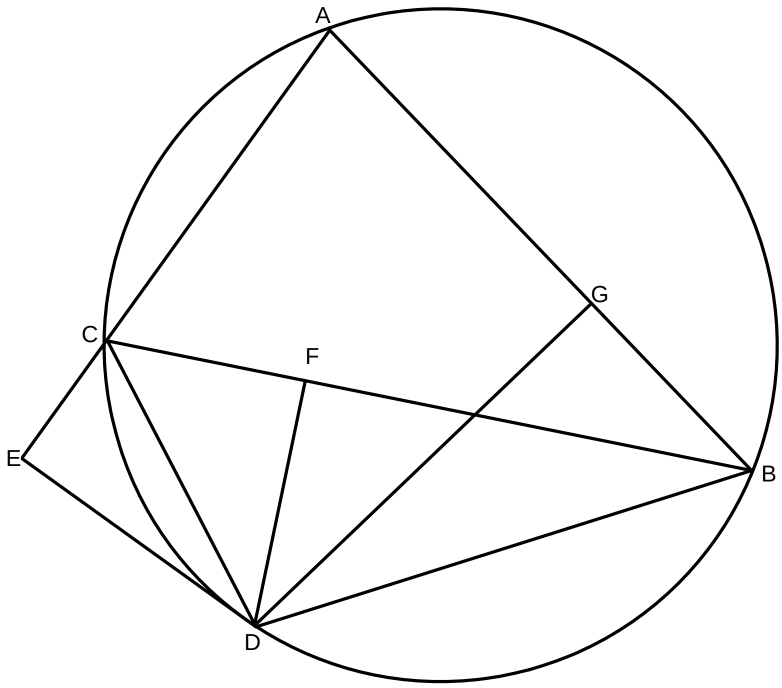
(iii) If $v = 1$, find the value of T and R correct to two decimal places. **2**

(iv) Find the maximum value of v in order that the particle remains in contact with the cone. **2**

(v) Deduce the maximum value that T can take if the particle is to remain in contact with the cone. **2**

Question 8 *(15 marks)* Use a SEPARATE writing booklet. **Marks**

(a)



The above diagram shows a triangle ABC inscribed in a circle with D a point on the arc BC. DE is perpendicular to AC produced, DF is perpendicular to BC and DG is perpendicular to AB.

Copy or trace this diagram into your writing booklet.

(i) Explain why $DECF$ and $DFGB$ are cyclic quadrilaterals. **2**

(ii) Show that the points E, F and G are collinear. **3**

Question 8 continues on the next page…

Question 8 continued…

Marks

(b) (i) Evaluate $\int_0^{\frac{\pi}{4}} \tan x \quad dx$. **2**

(ii) If $I_n = \int_0^{\frac{\pi}{4}} \tan^n x \quad dx$, n = 0, 1, 2, 3, …, show that for n = 2, 3, 4, … **3**

$$I_n = \frac{1}{n-1} - I_{n-2}$$

(iii) Hence, evaluate I_5. **2**

(c) If $P(x) = x^m\left(b^n - c^n\right) + b^m\left(c^n - x^n\right) + c^m\left(x^n - b^n\right)$ where m and n are positive integers, show that $x^2 - (b+c)x + bc$ is a factor of $P(x)$. **3**

End of paper

EXT 2 TRIAL HSC 2005

Q1

a) $\int x e^{-3x}\,dx$ OR Let $u = e^{-3x}$ $v' = x$

Let $u = x$ $v' = e^{-3x}$

$u' = 1$ $v = -\frac{1}{3}e^{-3x}$ 1

$\int uv'\,dx = uv - \int u'v\,dx$ /2

$= -\frac{1}{3}xe^{-3x} + \frac{1}{3}\int e^{-3x}\,dx$

$= -\frac{1}{3}xe^{-3x} - \frac{1}{9}e^{-3x} + C$ 1

b) $\int_0^{2/3} \sqrt{4-9x^2}\,dx$ Let $x = \frac{2}{3}\sin\alpha$

$\frac{dx}{d\alpha} = \frac{2}{3}\cos\alpha$.

$= \int_0^{\pi/2} \sqrt{4 - 9.\frac{4}{9}\sin^2\alpha}\,.\,\frac{2}{3}\cos\alpha\,d\alpha$

$dx = \frac{2}{3}\cos\alpha\,d\alpha$ 1

$= \frac{2}{3}\int_0^{\pi/2} 2\sqrt{\cos^2\alpha}\,.\cos\alpha\,d\alpha$. when $x=0$ $\alpha=0$

$x = 2/3$ $\alpha = \pi/2$.

$= \frac{4}{3}\int_0^{\pi/2} \cos^2\alpha\,d\alpha$.

$= \frac{4}{3}\int_0^{\pi/2} \frac{1}{2}(\cos 2\alpha + 1)\,d\alpha$ 1 /3

$= \frac{2}{3}\left[\frac{1}{2}\sin 2\alpha + \alpha\right]_0^{\pi/2}$

$= \frac{\pi}{3}$ 1

$\therefore \int_0^{2/3} \sqrt{4-9x^2}\,dx = \frac{\pi}{3}$.

c) $\displaystyle\int \frac{dx}{\sqrt{x^2-4x+29}}$

$\displaystyle = \int \frac{dx}{\sqrt{(x-2)^2+25}}$ 1 $\qquad$ /2

$= \ln\left(x-2+\sqrt{x^2-4x+29}\right)$ 1

d) $\displaystyle\int_4^6 \frac{2\,dt}{(t-1)(t-3)}$

$\displaystyle\frac{2}{(t-1)(t-3)} = \frac{A}{t-1} + \frac{B}{t-3}$ 1

$2 = A(t-3) + B(t-1)$

Let $t=1$ $\qquad$ $t=3$ $\qquad$ 1

$2=-2A$ $\qquad$ $2=2B$

$A=-1$ $\qquad$ $B=1$ $\qquad$ /4

$\displaystyle\therefore \int_4^6 \frac{2\,dt}{(t-1)(t-3)} = \int_4^6 \frac{-1}{t-1} + \frac{1}{t-3}\,dt$

$\displaystyle = \left[\ln\left(\frac{t-3}{t-1}\right)\right]_4^6$ 1

$\displaystyle = \ln\frac{3}{5} - \ln\frac{1}{3}$

$\displaystyle = \ln\frac{9}{5}$ 1

e) when $t=\tan\frac{x}{2}$, $\cos x = \dfrac{1-t^2}{1+t^2}$ 1

$\displaystyle\frac{dt}{dx} = \frac{1}{2}\sec^2\frac{x}{2}$

$\displaystyle = \frac{1}{2}\left(1+\tan^2\frac{x}{2}\right)$

$dt = \frac{1}{2}(1+t^2)\,dx$ 1

$\therefore dx = 2\,\frac{1}{1+t^2}\,dt$

$\displaystyle\int_0^{\pi/2} \frac{dx}{5+4\cos x} = \int_0^1 \frac{1}{5+4\left(\frac{1-t^2}{1+t^2}\right)} \cdot 2\,\frac{1}{(1+t^2)}\,dt$

$\displaystyle = 2\int_0^1 \frac{1}{5+5t^2+4-4t^2}\,dt$ $\qquad$ /4

$\displaystyle = 2\int_0^1 \frac{1}{9+t^2}\,dt$ 1

$\displaystyle = \frac{2}{3}\left[\tan^{-1}\frac{t}{3}\right]_0^1$ 1

$\displaystyle = \frac{2}{3}\tan^{-1}\left(\frac{1}{3}\right)$

Q2

a) i) $\bar{z} = 1+\sqrt{3}i$ (1) ans

ii) $z^2 = (1-\sqrt{3}i)^2$

$= 1-2\sqrt{3}i + 3i^2$

$= -2-2\sqrt{3}i$ (1) ans

iii) $\frac{1}{z} = \frac{1}{1-\sqrt{3}i} \times \frac{1+\sqrt{3}i}{1+\sqrt{3}i}$

$= \frac{1+\sqrt{3}i}{1+3}$

$= \frac{1}{4}+\frac{\sqrt{3}}{4}i$ (1) ans

b) $|z_1| = \sqrt{5}$ $\quad |z_2| = \sqrt{5}$

$\arg z_1 = \tan^{-1}(2)$ $\quad \arg z_2 = \tan^{-1}\left(\frac{1}{2}\right)$

i) $|z| = \frac{|z_1|}{|z_2|}$

$= 1$ (1) ans

ii) 

$\arg(z) = \arg(z_1) - \arg(z_2)$

$= \tan^{-1}(2) - \tan^{-1}\left(\frac{1}{2}\right)$

Let $\alpha = \tan^{-1} 2$ $\quad \beta = \tan^{-1}\frac{1}{2}$

$\therefore 2 = \tan\alpha$ $\quad \tan\beta = \frac{1}{2}$ (1) working

$\tan(\arg(z)) = \frac{2-\frac{1}{2}}{1-2.\frac{1}{2}}$

$= \frac{1\frac{1}{2}}{0}$

$\therefore \arg z = -\pi/2$. (1) ans

c) Let $z = a + ib$ where $z \in \mathbb{C}$ and $a, b \in \mathbb{R}$

LHS

$|z| = \sqrt{a^2 + b^2}$

$\therefore |z|^2 = a^2 + b^2$ ① LHS

RHS

$z\bar{z} = (a + ib)(a - ib)$

$= a^2 + aib - aib - i^2 b^2$

$= a^2 + b^2$

$\therefore |z|^2 = z\bar{z} \quad \forall z \in \mathbb{C}$. ① RHS

d) i) $\omega^3 = 1$

$\omega^3 - 1 = 0$

$\therefore (\omega - 1)(1 + \omega + \omega^2) = 0$

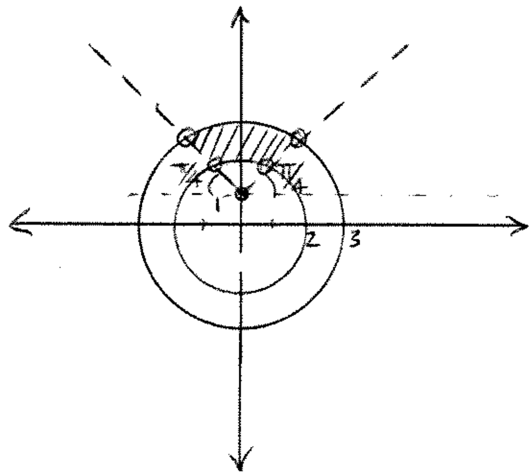
$\therefore 1 + \omega + \omega^2 = 0$ if $\omega \neq 1$. ① statement of condition.

$= 3$ if $\omega = 1$

ii) $\omega^4 + \omega^5 + \omega^6 = \omega^4(1 + \omega + \omega^2)$

$= 0$ or 3ω? ①

e)



① circular region w/ radii + solid lines

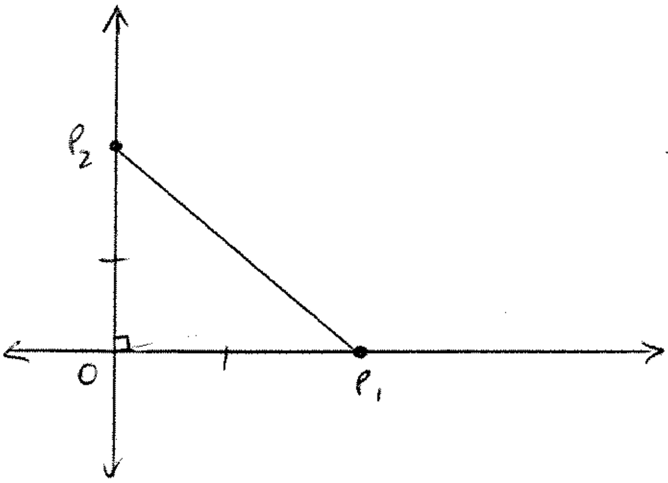
① angular region begins at (0, 1) with correct angles + dotted lines

OR

① Right idea

② Right idea with detail.

f)



① Diagram

$$|z_1| = |z_2|$$

$$\arg\left(\frac{z_1}{z_2}\right) = \frac{\pi}{2} \quad . \quad \therefore \quad z_2 = i z_1$$

$$\therefore \quad |z_1|^2 + |z_2|^2 = |z_1 - z_2|^2$$

Let $z_1 = a$ and $z_2 = ai$

Then $\quad z_1^2 + z_2^2 = a^2 - a^2$

$\qquad = 0$. ① specific case

Similarly, in the general case

$$z_2 = i z_1 .$$

$$\therefore \quad z_1^2 + z_2^2 = z_1^2 + (i z_1)^2$$ ① general case.

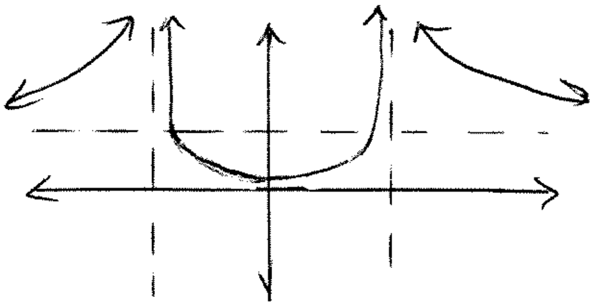
$$= z_1^2 - z_1^2$$

$$= 0 .$$

Q3

a)

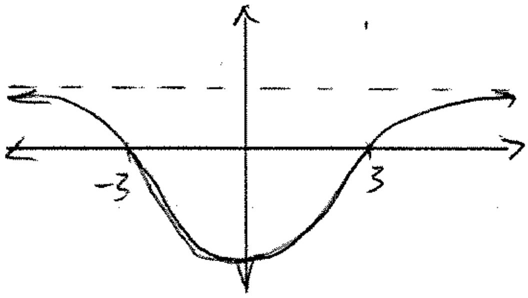
i/



① Branches less steep

① Centre section reflected
(should also be flatter
but no marks deducted
for this)

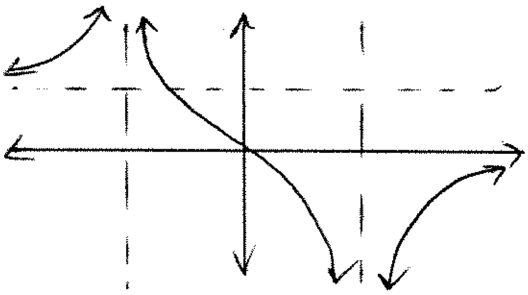
ii/



① Intercepts at 3, -3.

① Shape

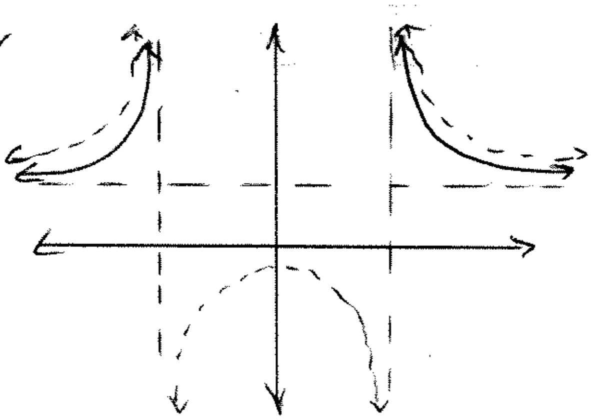
iii/



① Signs

① Shape

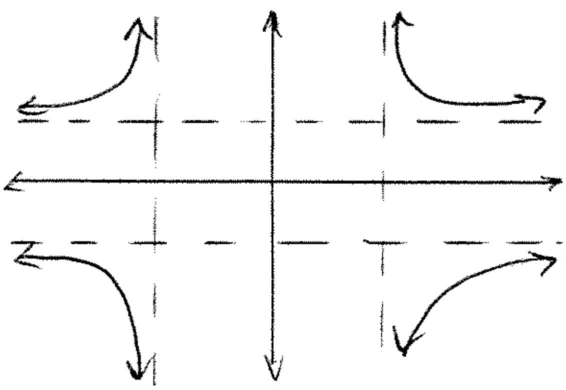
iv/



① Branches below original

① No $y = \sqrt{f(x)}$ for $f(x) < 0$.

v/



① Reflection over y of some section

① Correctly reflects only +ve y onto
-y axis

b) i) $3x^2 + y^2 - 2xy - 8x + 2 = 0$ ① chain rule

$6x + 2y\frac{dy}{dx} - 2x\frac{dy}{dx} - 2y - 8 = 0$

$\therefore \frac{dy}{dx}2(y-x) = -6x + 2y + 8$

① otherwise correct.

$\therefore \frac{dy}{dx} = \frac{3x-y-4}{x-y}$.

ii) $m = 2$

$\therefore \frac{3x-y-4}{x-y} = 2$

$3x - y - 4 = 2x - 2y$

$x + y - 4 = 0$

$\therefore y = 4 - x$ ① Condition on x + y

$\therefore 3x^2 + (4-x)^2 - 2x(4-x) - 8x + 2 = 0$

$3x^2 + 16 - 8x + x^2 - 8x + 2x^2 - 8x + 2 = 0$

$6x^2 - 24x + 18 = 0$

$x^2 - 4x + 3 = 0$

$(x-3)(x-1) = 0$ ① Solution

$\therefore x = 1$ or 3

$y = 3$ or 1

① Coordinates

$\therefore$ Coords of pts are $(1,3)$ + $(3,1)$.

Q4

a)

i/ $\alpha\beta = 4$

$\therefore \gamma = -4$

$b = 4 + \alpha\gamma + \beta\gamma$

$= 4 - 4(\alpha+\beta)$

Now $\alpha + \beta + \gamma = -2$. 1 Reasonable logic

$\therefore \alpha + \beta = -6$

$\therefore b = 4 - 24$

$\therefore b = -20$

ii/ Let ρ represent new roots + x the original roots

$\rho = x^2$

$\therefore x = \sqrt{\rho}$

$\therefore$ New polynomial has

$(\sqrt{\rho})^3 + 2(\sqrt{\rho})^2 - 20\sqrt{\rho} - 16 = 0$ 1 subst $\sqrt{\ }$

$\sqrt{\rho}(\rho - 20) = -2\rho + 16$.

$\rho(\rho-20)^2 = (16-2\rho)^2$ 1 square + simplify

$\rho(\rho^2 - 40\rho + 400) = 256 - 64\rho + 4\rho^2$

$\rho^3 - 44\rho^2 + 464\rho - 256 = 0$ is new polynomial.

b) i/ $P(\alpha) = 0$ since $P(\alpha) = (\alpha-\alpha)^3 Q(x)$

$= 0$

$P'(x) = 3(x-\alpha)^2 Q(x) + Q'(x)(x-\alpha)^3$

$(x-\alpha)^2(3Q(x) + (x-\alpha)Q'(x))$ 1 $P'(\alpha)$

$\therefore P'(\alpha) = 0$

$P''(\alpha) = 2(x-\alpha)(3Q(x) + (x-\alpha)Q'(x)) + (x-\alpha)^2(\ldots)$

$\therefore P''(\alpha) = 0$ 1 $P''(\alpha)$

$\therefore P(\alpha) = P'(\alpha) = P''(\alpha) = 0$.

ii/ $P(x) = 8x^4 - 25x^3 + 27x^2 - 11x + 1$

$P'(x) = 32x^3 - 75x^2 + 54x - 11$

$P''(x) = 96x^2 - 150x + 54$

Let $P''(x) = 0$

Then $96x^2 - 150x + 54 = 0$

$16x^2 - 25x + 9 = 0$

$(16x - 9)(x - 1) = 0$

$x = 9/16$ or $x = 1$ ① Identify triple root

Check $P(1) = 0$ $P'(1) =$

$\therefore P(x) = (x-1)^3(8x-1)$ ① Determine fourth root

$\therefore x = 1, 1/8$.

c) i/ $z^9 - 1 = 0$

$(z^3 - 1)(z^6 + z^3 + 1) = 0$ ①

Check

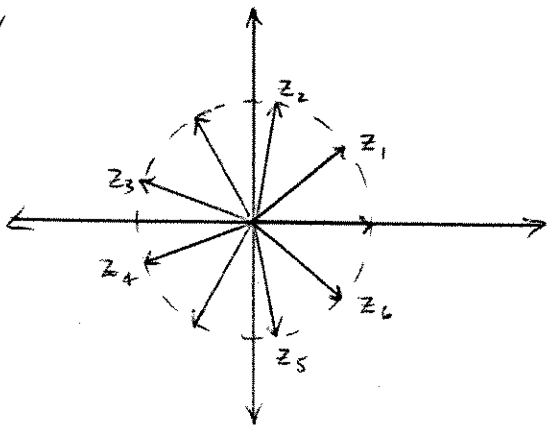
$z^9 + z^6 + z^3 - z^6 - z^3 - 1 = 0$ ① Working

$z^9 - 1 = 0$

$\therefore$ Solns of $z^6 + z^3 + 1 = 0$ are contained in $z^9 - 1 = 0$

ii/

iii/



Roots of $z^3 - 1 = 0$ are $cis\frac{2\pi}{3}$, $cis\frac{4\pi}{3}$, 1.

① Unit circle

① 9 roots marked w/ vectors at equal angles.

① Identify roots of $z^3 - 1$

① Mark all 6 roots of $z^6 + z^3 + 1 = 0$.

iv/ Roots are: $cis\left(\pm\frac{2\pi}{9}\right)$, $cis\left(\pm\frac{8\pi}{9}\right)$, $cis\left(\pm\frac{4\pi}{9}\right)$ ① Name roots

$\therefore$ Sum $= 2\cos\frac{2\pi}{9} + 2\cos\frac{8\pi}{9} + 2\cos\frac{4\pi}{9}$

but $\cos\frac{8\pi}{9} = -\cos\frac{\pi}{9}$ ① Identity on cos.

$\therefore$ Sum $= 2\left(\cos\frac{2\pi}{9} + \cos\frac{4\pi}{9} - \cos\frac{\pi}{9}\right)$

Ext 2 05

5

(a)(i) $b^2 = a^2(1-e^2)$ ①

$4 = 9(1-e^2)$

$e = \frac{\sqrt{5}}{3}$ ①

$S(\sqrt{5}, 0)$ ①

(ii) $x = \frac{9}{\sqrt{5}}$ ①

(iii) $S(0, \sqrt{5})$, $y = \frac{9}{\sqrt{5}}$ ① both

(b)(i) $\frac{2x}{16} - \frac{2y \frac{dy}{dx}}{9} = 0$ ①

$\frac{dy}{dx} = \frac{9x}{16y}$

$= \frac{3\sec\theta}{4\tan\theta}$ ①

tangent: $y - 3\tan\theta = \frac{3\sec\theta}{4\tan\theta}(x - 4\sec\theta)$ ①

etc.

normal: $y - 3\tan\theta = -\frac{4\tan\theta}{3\sec\theta}(x - 4\sec\theta)$ ①

(ii) At T, $x = 0$ $\therefore$ $y = \frac{-3}{\tan\theta} = -3\cot\theta$ ① correct intermediate step

At N, $x = 0$, $\therefore$ $y = \frac{16\tan\theta}{3} + 3\tan\theta = \frac{25\tan\theta}{3}$ ① "

(iii) $b^2 = a^2(e^2 - 1)$

$9 = 16(e^2 - 1)$

$e = \frac{5}{4}$

$\therefore S = (5, 0)$

$m_{NS} \cdot m_{TS} = \frac{\frac{25}{3}\tan\theta}{-5} \cdot \frac{-3\cot\theta}{-5} = -1$ $\therefore N\hat{S}T = 90^\circ$

$\therefore$ circle with diameter NT passes through S ($N\hat{S}T$ = angle in semicircle)

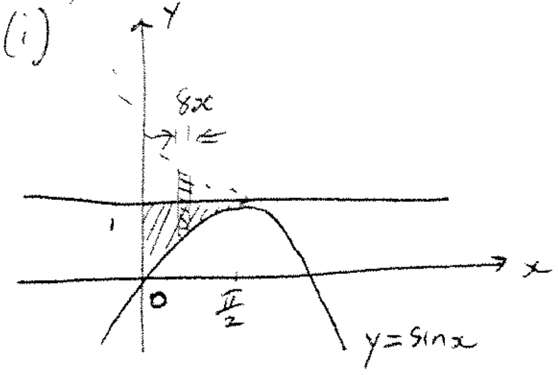
① correct focus

② 1 for each gradient

① stating '∠ in semicircle'

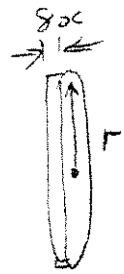
5) (a)

(i)



1

(ii)



$\delta V \doteqdot \pi r^2 \delta x$

$V \doteqdot \sum_{x=0}^{x=\frac{\pi}{2}} \pi (1-y)^2 \delta x$

$= \lim_{\delta x \to 0} \sum_{x=0}^{x=\frac{\pi}{2}} \pi (1-y)^2 \delta x$

$V = \pi \int_0^{\frac{\pi}{2}} (1-y)^2 dx$ 1

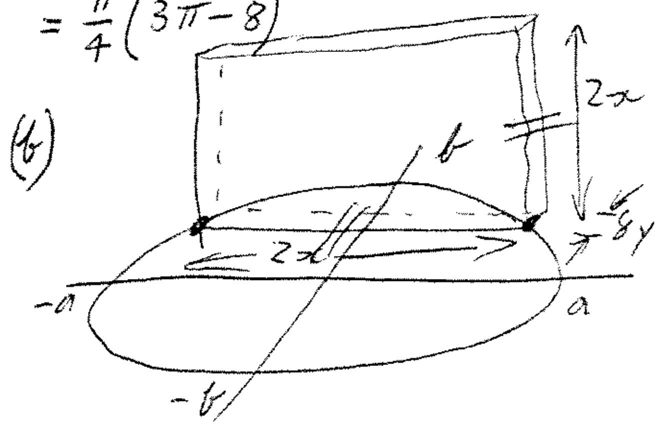
$= \pi \int_0^{\frac{\pi}{2}} (1-2y+y^2)\, dx \ , \ y = \sin x$

$= \pi \int_0^{\frac{\pi}{2}} \left(1 - 2\sin x + \frac{1}{2}(1-\cos 2x)\right) dx$

$= \pi \left[x + 2\cos x + \frac{1}{2}\left(x - \frac{1}{2}\sin 2x\right) \right]_0^{\frac{\pi}{2}}$ 1

$= \frac{\pi}{4}(3\pi - 8)$ 1

(b)



$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

$x^2 = a^2\left(1 - \frac{y^2}{b^2}\right)$

$\delta V \doteqdot 4x^2 \delta y$

$V \doteqdot \sum_{y=-b}^{y=b} 4x^2 \delta y$

$V = \lim_{\delta y \to 0} \sum_{y=-b}^{y=b} 4x^2 \delta y$

6 (b) ctd

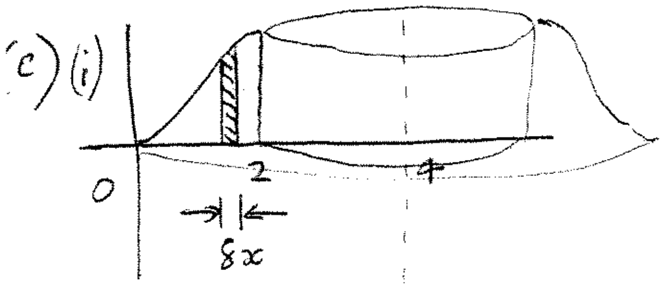
$$V = \int_{-b}^{b} 4a^2\left(1-\frac{y^2}{b^2}\right)dy \qquad 1$$

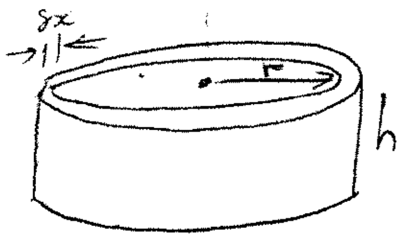
$$= 8a^2\int_{0}^{b}\left(1-\frac{y^2}{b^2}\right)dy$$

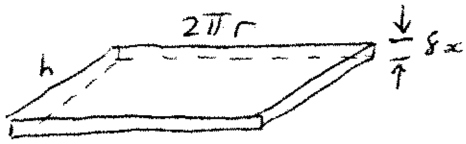
$$= 8a^2\left[y-\frac{y^3}{3b^2}\right]_0^b \qquad 1$$

$$= \frac{16a^2b}{3} \qquad 1$$

(c) (i)







$$\delta V \doteqdot 2\pi r h\,\delta x \qquad 1$$

$$= 2\pi(4-x)\left(\frac{x^2}{x^2+1}\right) \qquad 1$$

$$= 2\pi(4-x)\left(\frac{x^2+1-1}{x^2+1}\right) \qquad 1$$

$$= 2\pi(4-x)\left(1-\frac{1}{1+x^2}\right)$$

(ii) $$V = \int_0^2 2\pi(4-x)\left(1-\frac{1}{1+x^2}\right)dx$$

$$= 2\pi\int_0^2\left(4-\frac{4}{1+x^2}-x+\frac{x}{1+x^2}\right)dx$$

$$= 2\pi\left[4x-4\tan^{-1}x-\frac{x^2}{2}+\frac{1}{2}\log(1+x^2)\right]_0^2$$

$$= 2\pi\left(6-4\tan^{-1}2+\frac{1}{2}\log 5\right)$$

7) (a) (i) $m\ddot{x} = -mk(1+v^2)$

$\ddot{x} = -k(1+v^2)$ (1)

$\frac{dv}{dt} = -k(1+v^2)$

$\frac{dt}{dv} = -\frac{1}{k} \cdot \frac{1}{1+v^2}$ (1)

$\int_0^T dt = -\frac{1}{k}\int_U^0 \frac{dv}{1+v^2}$

$T = \frac{1}{k}\left[\tan^{-1} v\right]_0^U$ (1)

$= \frac{1}{k}\tan^{-1} U$

(ii) $v\frac{dv}{dx} = -k(1+v^2)$ (1)

$\frac{dv}{dx} = -k \cdot \frac{1+v^2}{v}$

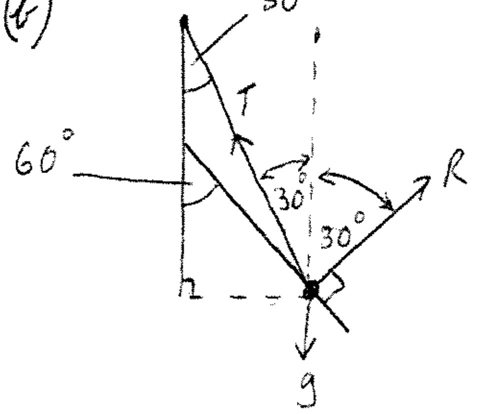
$\frac{dx}{dv} = -\frac{1}{k} \cdot \frac{v}{1+v^2}$

$\int_0^X dx = -\frac{1}{k}\int_U^0 \frac{v\,dv}{1+v^2}$

$X = +\frac{1}{2k}\left[\log(1+v^2)\right]_0^U$

$= \frac{1}{2k}\log(1+U^2)$ (1)

(b)



(1) Forces

(1) Angles

7 ctd

Horizontal: $T\sin 30^\circ - R\sin 30^\circ = \frac{mv^2}{r}$ $m=1, r=0.5$ 1

$T - R = 4v^2$ —— 1

Vertical: $T\cos 30^\circ + R\cos 30^\circ = mg$ 1

$T + R = \frac{2g}{\sqrt{3}}$ —— 2

iii) $v = 1$ $\therefore T - R = 4$ —— 3

2 + 3

$2T = \frac{2g}{\sqrt{3}} + 4$

$T = \frac{g}{\sqrt{3}} + 2$, $g = 9.8$

≈ 7.66 (2 decimal places) 1

sub in 3

$R \approx 3.66$ (2 decimal places) 1

iv) T_{max} when $R = 0$ 1 $R=0$ or $R>0$

$\therefore T = \frac{2g}{\sqrt{3}}$ is the upper limit for T

1 Answer

(or $R = \frac{2g}{\sqrt{3}} - T$ $R > 0$ $\therefore \frac{2g}{\sqrt{3}} - T > 0$

$T < \frac{2g}{\sqrt{3}}$)

v) $R = 0 \Rightarrow 4v^2 = T = \frac{2g}{\sqrt{3}}$

$v^2 = \frac{g}{2\sqrt{3}}$ 1 v^2

$v = \left(\frac{g}{2\sqrt{3}}\right)^{\frac{1}{2}}$ 1 Answer.

or $R =$ ~~4R~~ $T - 4v^2 > 0$

$v^2 < \frac{T}{4}$, $v < \left(\frac{g}{2\sqrt{3}}\right)^{\frac{1}{2}}$

(8) (a) (i) $C\hat{E}F + C\hat{F}D = 90° + 90°$ (Given)

$= 180°$

$\therefore DECF$ cyclic (opp $\angle$s supplementary) (1)

$D\hat{F}B = D\hat{G}B = 90°$ (Given)

$\therefore DFGB$ cyclic (DB diameter, $D\hat{F}B, D\hat{G}B$ $\angle$s in semicircle) (1)

(ii) Let $E\hat{C}D = \alpha$

$\therefore E\hat{F}D = \alpha$ ($\angle$s in same segment of circle ECFD)

But $G\hat{B}D = E\hat{F}D$ (ext $\angle$ of cyclic quad. DFGB = opp. int. $\angle$)

$= \alpha$

$\therefore D\hat{F}G = 180° - \alpha$ (opp. supplementary $\angle$s of cyclic quad DFGB)

$\therefore E\hat{F}D + D\hat{F}G = \alpha + 180° - \alpha$

$= 180°$

$\therefore E, F, G$ collinear

(1) 2 appropriate facts with reasons.

(1) Using 3 cyclic quads

(1) correct answer.

(b) (i) $\int_0^{\frac{\pi}{4}} \tan x \, dx$

$= \int_0^{\frac{\pi}{4}} \frac{\sin x}{\cos x} \, dx$

$= -\left[\log(\cos x)\right]_0^{\frac{\pi}{4}}$ (1)

$= \log\sqrt{2}$ (1)

8) ctd

(ii) $$I_n = \int_0^{\frac{\pi}{4}} \tan^n x \, dx$$

$$= \int_0^{\frac{\pi}{4}} \tan^{n-2} x (\sec^2 x - 1) \, dx$$ ①

$$= \int_0^{\frac{\pi}{4}} \tan^{n-2} x . \sec^2 x \, dx - \int_0^{\frac{\pi}{4}} \tan^{n-2} x \, dx$$

$$= \frac{1}{n-1}\left[\tan^{n-1} x\right]_0^{\frac{\pi}{4}} - I_{n-2}$$ ② 1 each.

$$= \frac{1}{n-1} - I_{n-2}$$

iii) $$I_5 = \frac{1}{4} - I_3$$

$$= \frac{1}{4} - \left(\frac{1}{2} - I_1\right)$$

$$= -\frac{1}{4} + \log\sqrt{2}$$

(c) If $(x-b)$ and $(x-c)$ are factors then $(x-b)(x-c)$ is a factor ①

Note: $(x-b)(x-c) = x^2 - (b+c)x + bc$

$$P(b) = b^m(c^n - b^n) + b^m(b^n - c^n) + c^m(b^n - b^n)$$

$$= 0$$ ①

$$P(c) = c^m(b^n - c^n) + b^m(c^n - c^n) + c^m(c^n - b^n)$$

$$= 0$$ ①

$\therefore$ $(x-b)$, $(x-c)$ are factors.