Total marks – 120
Attempt Question 1-8
All questions are of equal value
Answer each question in the appropriate writing booklet. Extra writing booklets are available.

Question 1 (15 marks) **Marks**

(a) Find

(i) $\displaystyle\int \frac{\cos\theta}{\sin^5\theta}\,d\theta$ **2**

(ii) $\displaystyle\int \frac{dx}{x^2+2x+2}$ **2**

(b) Use the substitution $t=\tan\dfrac{x}{2}$ to find $\displaystyle\int \frac{dx}{5+4\cos x+3\sin x}$ **3**

(c) Use the substitution $u=-x$ to evaluate $\displaystyle\int_{-1}^{1} \frac{dx}{e^x+1}$ **3**

(d) Evaluate the following definite integrals:

(i) $\displaystyle\int_0^1 \cos^{-1}x\,dx$ **2**

(ii) $\displaystyle\int_1^2 x(\ln x)^2\,dx$ **3**

Question 2 (15 marks) Start a new booklet

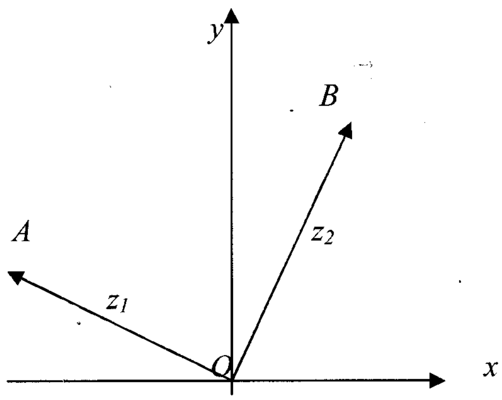
(a) If $z = 3 - 2i$, mark clearly on an Argand diagram the points represented by

(i) $2z$ **1**

(ii) $-2iz$ **2**

(b) If $|z_1 + z_2| = |z_1 - z_2|$, find the possible values of $\arg\left(\dfrac{z_1}{z_2}\right)$. **3**

(c)



In the Argand diagram, vectors $\overrightarrow{OA}$ and $\overrightarrow{OB}$ represent the complex numbers $z_1 = 2\left(\cos\dfrac{4\pi}{5} + i\sin\dfrac{4\pi}{5}\right)$ and $z_2 = 2\left(\cos\dfrac{7\pi}{15} + i\sin\dfrac{7\pi}{15}\right)$ respectively.

(i) Show that ΔOAB is equilateral **2**

(ii) Express $z_2 - z_1$ in modulus-argument form. **3**

(d) z is a complex number such that $\arg z = \dfrac{\pi}{3}$ and $|z| \le 2$.

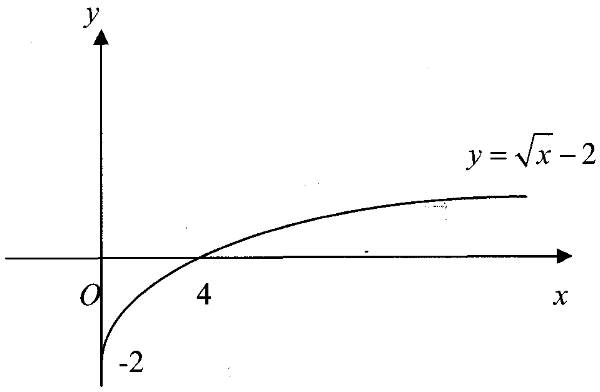
(i) Show the locus of the point P representing z in the Argand diagram. **2**

(ii) Find the possible values of the principal argument of $z - 1$ for z on this locus. **2**

Question 3 (15 marks) Start a new booklet

(a) Twelve different books are made into four parcels of three each. How many different sets of parcels could be made? **3**

(b)



The diagram shows the graph of the function $f(x)=\sqrt{x}-2$. On separate diagrams sketch the following graphs, showing clearly any intercepts on the coordinate axes and the equations of any asymptotes:

(i) $y=\left|f(x)\right|$ **1**

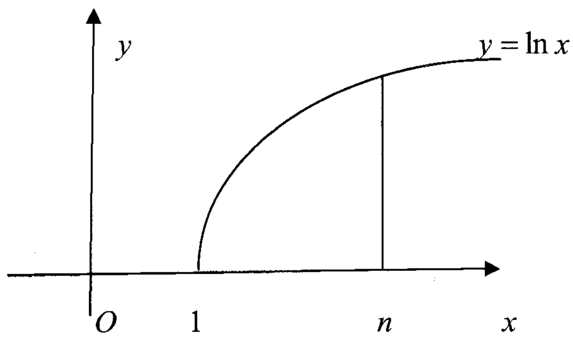
(ii) $y=\left[f(x)\right]^2$ **1**

(iii) $y=\dfrac{1}{f(x)}$ **2**

(iv) $y=\ln f(x)$ **2**

Question 3 Continued.

(c)



(i) Use the trapezoidal rule with n function values to approximate $\int_1^n \ln x \, dx$ **2**

(ii) Show that $\frac{d}{dx}(x \ln x - x) = \ln x$ and hence find the exact value of **2**

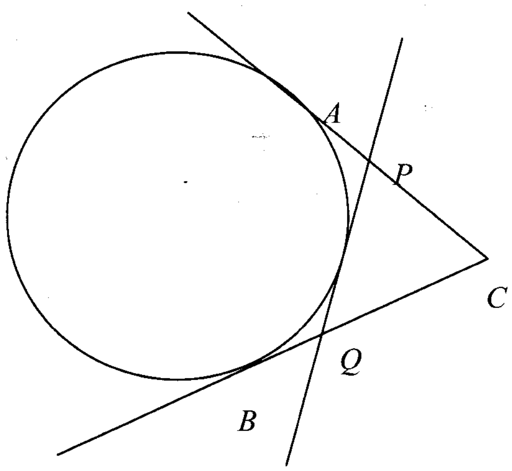
$\int_1^n \ln x \, dx$.

2

(iii) Deduce that $\ln n! < \left(n + \frac{1}{2}\right)\ln n - n + 1$

Question 4 (15 marks) Start a new booklet

(a) A and B are two points on a circle. Tangents at A and B meet at C. A third tangent cuts CA and CB in P and Q respectively, as shown in the diagram. Show that the perimeter of ΔCPQ is constant and independent of PQ. **3**



(b) The polynomial $P(x)$ leaves a remainder of 9 when divided by $(x-2)$ and a remainder of 4 when divided by $(x-3)$. Find the remainder when $P(x)$ is divided by $(x-2)(x-3)$. **4**

(c) $z=\cos\theta+i\sin\theta$

(i) Show that $z^n+z^{-n}=2\cos n\theta$ for $n=1,2,3,...$ **2**

(ii) Hence show that $4\cos\theta\cos 2\theta\cos 3\theta=1+\cos 2\theta+\cos 4\theta+\cos 6\theta$. **3**

(iii) Hence, solve $\cos^2\theta+\cos^2 2\theta+\cos^2 3\theta=1$, giving general solutions. **3**

Question 5 (15 marks) Start a new booklet

(a) $P\left(3p,\dfrac{3}{p}\right)$ and $Q\left(3q,\dfrac{3}{q}\right)$ are points on different branches of the hyperbola $xy=9$.

(i) Find the equation of the tangent at P. **2**

(ii) Find the point of intersection, T, of the tangents at P and Q. **2**

(iii) If the chord PQ passes through the point $(0,2)$, find the locus of T, **3**

(iv) Find the restriction on the locus of T. **1**

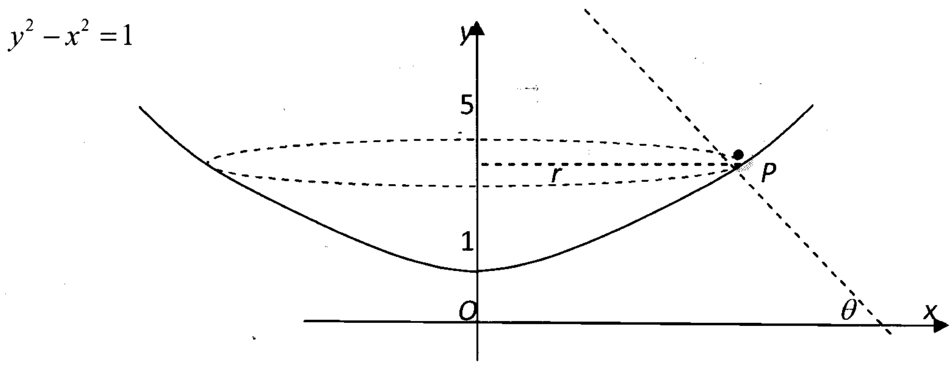
(b) The region bounded by the graphs of $y=x^2$ and $y=x+2$ is revolved around the line $x=3$. Express the volume of the resulting solid as a definite integral. Do not calculate the value of this integral. **3**

(c) A solid has, as its base, the circular region in the xy-plane bounded by the graph of $x^2+y^2=a^2$, where $a>0$. Find the volume of the solid if every cross-section by a plane perpendicular to the x-axis is an equilateral triangle with one side in the base. **4**

Question 6 (15 marks) Start a new booklet

(a) A particle of mass m moves in a straight line away from a fixed point O in the line, such that at time t its displacement from O is x and its velocity is v. At time $t=0$, $x=1$ and $v=0$. Subsequently, the only force acting on the particle is one of magnitude $m\dfrac{k}{x^2}$, where k is a positive constant in a direction away from O. Show that v cannot exceed $\sqrt{(2k)}$. **4**

(b)



A bowl is formed by rotating the hyperbola $y^2 - x^2 = 1$ for $1 \le y \le 5$ about the y axis. A particle P of mass m moves around the inner surface of the bowl in a horizontal circle with constant angular velocity ω.

(i) Show that if the radius of the circle in which P moves is r, then the normal to the surface at P makes an angle θ with the horizontal where $\tan\theta = \dfrac{\sqrt{1+r^2}}{r}$. **4**

(ii) Draw a diagram showing the forces acting on P. **1**

(iii) Find expressions for the radius r of the circle of motion and the magnitude of the reaction force between the surface and the particle in terms of m, g and ω. **3**

(iv) Find the values of ω for which the described motion of P is possible. **3**

Question 7 (15 marks) Start a new booklet

(a) The ellipse E : $\left(\dfrac{x}{5}\right)^2+\left(\dfrac{y}{3}\right)^2=1$ has foci $S(4,0)$ and $S'(-4,0)$.

(i) Sketch the ellipse E indicating its foci S, S' and its directrices. **1**

(ii) Show that the tangent at $P(x_1,y_1)$ on the ellipse E has equation **1**

$9xx_1+25yy_1=225$.

(iii) The line joining $P(x_1,y_1)$ to $Q(x_2,y_2)$ passes through S. Show that **2**

$4(y_2-y_1)=x_1y_2-x_2y_1$.

(iv) It is also known that $Q(x_2,y_2)$ lies on E. Show that the tangents at **2**

P and Q on the ellipse intersect on the directrix corresponding to S.

(v) Find the equation of the normal to E at P and decide under what **1**

circumstances, if any, it passes through S or S'.

(b) $I_n=\displaystyle\int_1^e(1-\ln x)^n\,dx$, $n=1,2,3,\ldots$

(i) Show $I_n=-1+nI_{n-1}$, $n=1,2,3,\ldots$ **2**

(ii) Hence evaluate $\displaystyle\int_1^e(1-\ln x)^3\,dx$. **1**

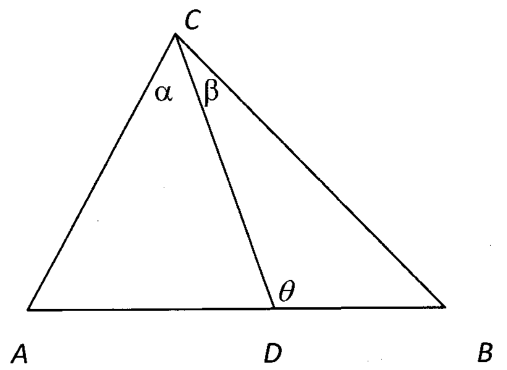
(iii) Show that $\dfrac{I_n}{n!}=e-\displaystyle\sum_{r=0}^{n}\frac{1}{r!}$, $n=1,2,3,\ldots$ **2**

(iv) Show that $0\le I_n\le e-1$. **1**

(v) Deduce that $\displaystyle\lim_{n\to\infty}\sum_{r=0}^{n}\frac{1}{r!}=e$. **2**

Question 8 (15 marks) Start a new booklet

(a)



In ΔABC, D is the point on AB that divides AB internally in the ratio $m : n$.
$\angle ACD = \alpha$, $\angle BCD = \beta$ and $\angle CDB = \theta$.

(i) By using the sine rule in each of ΔCAD and ΔCDB, show that **4**

$$\frac{\sin(\theta+\beta)\sin\alpha}{\sin(\theta-\alpha)\sin\beta} = \frac{m}{n}.$$

(ii) Hence show that $\tan\theta = \dfrac{(m+n)\tan\alpha\tan\beta}{m\tan\beta - n\tan\alpha}$. **3**

(b) Let $f(x)$ be a function which satisfies the equation:

$f(xy) = f(x) + f(y)$ for all $x, y \neq 0$.

(i) Show that $f(1) = 0 = f(-1)$ and that $f(x)$ is an even function. **2**

(ii) Prove that $f(x+y) - f(x) = f\left(1+\dfrac{y}{x}\right)$ for $x, y, x+y \neq 0$ **2**

(iii) Suppose $f(x)$ is differentiable at $x = 1$ and $f'(1) = 1$. Deduce that $f(x)$ is differentiable at any $x \neq 0$ and $f'(x) = \dfrac{1}{x}$. **4**

End of Examination