

**Total marks – 120****Attempt Question 1-8****All questions are of equal value**

Answer each question in the appropriate writing booklet. Extra writing booklets are available.

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**Question 1** (15 marks)**Marks**

(a) Find

(i)  $\int \frac{\cos \theta}{\sin^5 \theta} d\theta$  **2**

(ii)  $\int \frac{dx}{x^2 + 2x + 2}$  **2**

(b) Use the substitution  $t = \tan \frac{x}{2}$  to find  $\int \frac{dx}{5 + 4 \cos x + 3 \sin x}$  **3**

(c) Use the substitution  $u = -x$  to evaluate  $\int_{-1}^1 \frac{dx}{e^x + 1}$  **3**

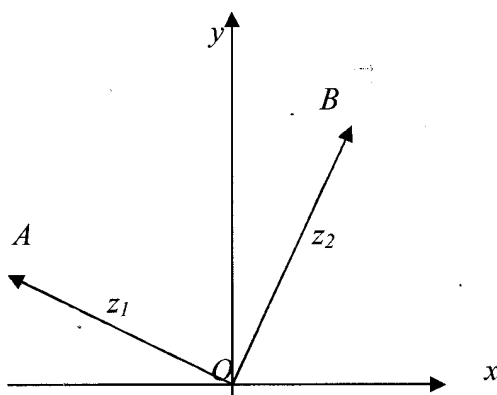
(d) Evaluate the following definite integrals:

(i)  $\int_0^1 \cos^{-1} x dx$  **2**

(ii)  $\int_1^2 x(\ln x)^2 dx$  **3**

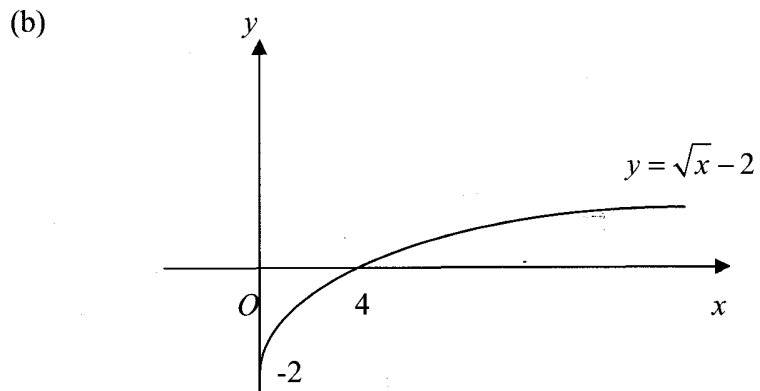
**Question 2** (15 marks) Start a new booklet(a) If  $z = 3 - 2i$ , mark clearly on an Argand diagram the points represented by(i)  $2z$  1(ii)  $-2iz$  2(b) If  $|z_1 + z_2| = |z_1 - z_2|$ , find the possible values of  $\arg\left(\frac{z_1}{z_2}\right)$ . 3

(c)

In the Argand diagram, vectors  $\overline{OA}$  and  $\overline{OB}$  represent the complex numbers $z_1 = 2\left(\cos\frac{4\pi}{5} + i\sin\frac{4\pi}{5}\right)$  and  $z_2 = 2\left(\cos\frac{7\pi}{15} + i\sin\frac{7\pi}{15}\right)$  respectively.(i) Show that  $\triangle OAB$  is equilateral 2(ii) Express  $z_2 - z_1$  in modulus-argument form. 3(d)  $z$  is a complex number such that  $\arg z = \frac{\pi}{3}$  and  $|z| \leq 2$ .(i) Show the locus of the point  $P$  representing  $z$  in the Argand diagram. 2(ii) Find the possible values of the principal argument of  $z - 1$  for  $z$  on this locus. 2

**Question 3** (15 marks) Start a new booklet

- (a) Twelve different books are made into four parcels of three each. How many different sets of parcels could be made? **3**

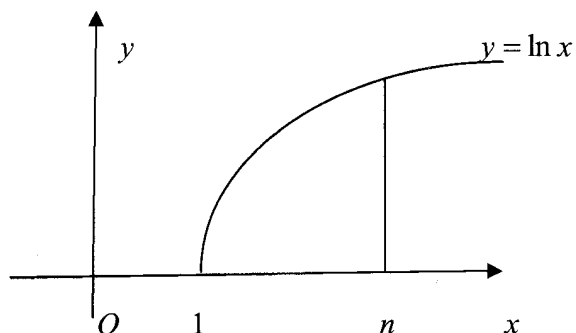


The diagram shows the graph of the function  $f(x) = \sqrt{x} - 2$ . On separate diagrams sketch the following graphs, showing clearly any intercepts on the coordinate axes and the equations of any asymptotes:

- (i)  $y = |f(x)|$  **1**
- (ii)  $y = [f(x)]^2$  **1**
- (iii)  $y = \frac{1}{f(x)}$  **2**
- (iv)  $y = \ln f(x)$  **2**

**Question 3 Continued.**

(c)



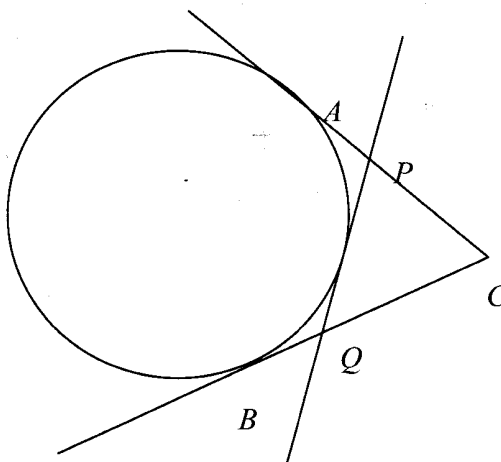
(i) Use the trapezoidal rule with  $n$  function values to approximate  $\int_1^n \ln x \, dx$  2

(ii) Show that  $\frac{d}{dx}(x \ln x - x) = \ln x$  and hence find the exact value of  $\int_1^n \ln x \, dx$ . 2

(iii) Deduce that  $\ln n! < \left(n + \frac{1}{2}\right) \ln n - n + 1$  2

**Question 4** (15 marks) Start a new booklet

- (a)  $A$  and  $B$  are two points on a circle. Tangents at  $A$  and  $B$  meet at  $C$ . A third tangent cuts  $CA$  and  $CB$  in  $P$  and  $Q$  respectively, as shown in the diagram. Show that the perimeter of  $\triangle CPQ$  is constant and independent of  $PQ$ . **3**



- (b) The polynomial  $P(x)$  leaves a remainder of 9 when divided by  $(x-2)$  and a remainder of 4 when divided by  $(x-3)$ . Find the remainder when  $P(x)$  is divided by  $(x-2)(x-3)$ . **4**

(c)  $z = \cos \theta + i \sin \theta$

- (i) Show that  $z^n + z^{-n} = 2 \cos n\theta$  for  $n = 1, 2, 3, \dots$  **2**
- (ii) Hence show that  $4 \cos \theta \cos 2\theta \cos 3\theta = 1 + \cos 2\theta + \cos 4\theta + \cos 6\theta$ . **3**
- (iii) Hence, solve  $\cos^2 \theta + \cos^2 2\theta + \cos^2 3\theta = 1$ , giving general solutions. **3**

**Question 5** (15 marks) Start a new booklet

(a)  $P\left(3p, \frac{3}{p}\right)$  and  $Q\left(3q, \frac{3}{q}\right)$  are points on different branches of the hyperbola  $xy = 9$ .

(i) Find the equation of the tangent at  $P$ .

2

(ii) Find the point of intersection,  $T$ , of the tangents at  $P$  and  $Q$ .

2

(iii) If the chord  $PQ$  passes through the point  $(0, 2)$ , find the locus of  $T$ ,

3

(iv) Find the restriction on the locus of  $T$ .

1

(b) The region bounded by the graphs of  $y = x^2$  and  $y = x + 2$  is revolved around the line  $x = 3$ . Express the volume of the resulting solid as a definite integral. Do not calculate the value of this integral.

3

(c) A solid has, as its base, the circular region in the  $xy$ -plane bounded by the graph of  $x^2 + y^2 = a^2$ , where  $a > 0$ . Find the volume of the solid if every cross-section by a plane perpendicular to the  $x$ -axis is an equilateral triangle with one side in the base.

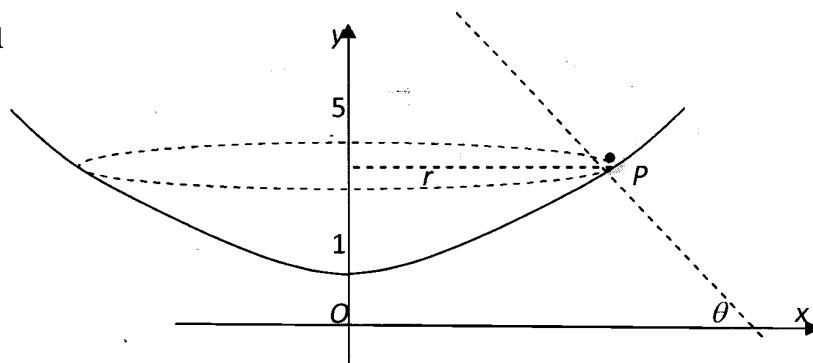
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**Question 6** (15 marks) Start a new booklet

- (a) A particle of mass  $m$  moves in a straight line away from a fixed point  $O$  in the line, such that at time  $t$  its displacement from  $O$  is  $x$  and its velocity is  $v$ . At time  $t = 0$ ,  $x = 1$  and  $v = 0$ . Subsequently, the only force acting on the particle is one of magnitude  $m \frac{k}{x^2}$ , where  $k$  is a positive constant in a direction away from  $O$ . Show that  $v$  cannot exceed  $\sqrt{(2k)}$ . 4

(b)

$$y^2 - x^2 = 1$$



A bowl is formed by rotating the hyperbola  $y^2 - x^2 = 1$  for  $1 \leq y \leq 5$  about the  $y$  axis. A particle  $P$  of mass  $m$  moves around the inner surface of the bowl in a horizontal circle with constant angular velocity  $\omega$

- (i) Show that if the radius of the circle in which  $P$  moves is  $r$ , then the normal to the surface at  $P$  makes an angle  $\theta$  with the horizontal 4  
 where  $\tan \theta = \frac{\sqrt{1+r^2}}{r}$ .
- (ii) Draw a diagram showing the forces acting on  $P$ . 1
- (iii) Find expressions for the radius  $r$  of the circle of motion and the magnitude of the reaction force between the surface and the particle in terms of  $m, g$  and  $\omega$ . 3
- (iv) Find the values of  $\omega$  for which the described motion of  $P$  is possible. 3

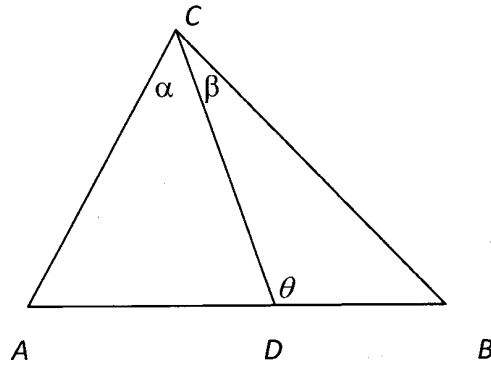
**Question 7** (15 marks) Start a new booklet

- (a) The ellipse  $E : \left(\frac{x}{5}\right)^2 + \left(\frac{y}{3}\right)^2 = 1$  has foci  $S(4,0)$  and  $S'(-4,0)$ . 1
- (i) Sketch the ellipse  $E$  indicating its foci  $S$ ,  $S'$  and its directrices. 1
- (ii) Show that the tangent at  $P(x_1, y_1)$  on the ellipse  $E$  has equation 1  
 $9xx_1 + 25yy_1 = 225$ .
- (iii) The line joining  $P(x_1, y_1)$  to  $Q(x_2, y_2)$  passes through  $S$ . Show that 2  
 $4(y_2 - y_1) = x_1y_2 - x_2y_1$ .
- (iv) It is also known that  $Q(x_2, y_2)$  lies on  $E$ . Show that the tangents at 2  
 $P$  and  $Q$  on the ellipse intersect on the directrix corresponding to  $S$ .
- (v) Find the equation of the normal to  $E$  at  $P$  and decide under what 1  
circumstances, if any, it passes through  $S$  or  $S'$ .
- (b)  $I_n = \int_1^e (1 - \ln x)^n dx$ ,  $n = 1, 2, 3, \dots$
- (i) Show  $I_n = -1 + nI_{n-1}$ ,  $n = 1, 2, 3, \dots$  2
- (ii) Hence evaluate  $\int_1^e (1 - \ln x)^3 dx$ . 1
- (iii) Show that  $\frac{I_n}{n!} = e - \sum_{r=0}^n \frac{1}{r!}$ ,  $n = 1, 2, 3, \dots$  2
- (iv) Show that  $0 \leq I_n \leq e - 1$ . 1
- (v) Deduce that  $\lim_{n \rightarrow \infty} \sum_{r=0}^n \frac{1}{r!} = e$ . 2



**Question 8** (15 marks) Start a new booklet

(a)



In  $\triangle ABC$ ,  $D$  is the point on  $AB$  that divides  $AB$  internally in the ratio  $m : n$ .  
 $\angle ACD = \alpha$ ,  $\angle BCD = \beta$  and  $\angle CDB = \theta$ .

(i) By using the sine rule in each of  $\triangle CAD$  and  $\triangle CDB$ , show that

$$\frac{\sin(\theta + \beta) \sin \alpha}{\sin(\theta - \alpha) \sin \beta} = \frac{m}{n}.$$

(ii) Hence show that  $\tan \theta = \frac{(m+n) \tan \alpha \tan \beta}{m \tan \beta - n \tan \alpha}$ .

(b) Let  $f(x)$  be a function which satisfies the equation:

$$f(xy) = f(x) + f(y) \text{ for all } x, y \neq 0.$$

(i) Show that  $f(1) = 0 = f(-1)$  and that  $f(x)$  is an even function.

(ii) Prove that  $f(x+y) - f(x) = f\left(1 + \frac{y}{x}\right)$  for  $x, y, x+y \neq 0$

(iii) Suppose  $f(x)$  is differentiable at  $x=1$  and  $f'(1) = 1$ . Deduce that

$$f(x) \text{ is differentiable at any } x \neq 0 \text{ and } f'(x) = \frac{1}{x}.$$

**End of Examination**