NORTH SYDNEY BOYS HIGH SCHOOL

100 Years 1912 ~ 2012

MATHEMATICS (EXTENSION 1)

2012 HSC Course Assessment Task 3 (Trial Examination)

June 27, 2012

General instructions

- Working time – 2 hours. (plus 5 minutes reading time)
- Write using blue or black pen. Where diagrams are to be sketched, these may be done in pencil.
- Board approved calculators may be used.
- Attempt **all** questions.
- At the conclusion of the examination, bundle the booklets + answer sheet used in the correct order within this paper and hand to examination supervisors.

SECTION I

- Mark your answers on the answer sheet provided (numbered as page 9)

SECTION II

- Commence each new question on a new page. Write on both sides of the paper.
- All necessary working should be shown in every question. Marks may be deducted for illegible or incomplete working.

STUDENT NUMBER: .... **# BOOKLETS USED**: .....

Class (please ✔)

- ◯ 12M4A – Mr Weiss
- ◯ 12M4B – Mr Ireland
- ◯ 12M4C – Mr Fletcher
- ◯ 12M3C – Ms Ziaziaris
- ◯ 12M3D – Mr Lowe
- ◯ 12M3E – Mr Lam

Marker's use only.

QUESTION	1-10	11	12	13	14	Total	%
MARKS	/10	/15	/15	/15	/15	/70	

Section I: Objective response

Mark your answers on the multiple choice sheet provided. **Marks**

1. Which is the correct value of $\displaystyle\lim_{x\to 0}\frac{3x}{\sin 2x}$? **1**

(A) 0 (B) $\frac{2}{3}$ (C) $\frac{3}{2}$ (D) 3

2. Which of the following is the acute angle (correct to the nearest degree) between the two lines $2x - y + 1 = 0$ and $3x + y - 4 = 0$? **1**

(A) 11° (B) 45° (C) 79° (D) 135°

3. Which of the following expressions will result in the coordinates of the point P which divides the interval AB externally in the ratio $3:2$, given A is $(-5, 2)$ and B is $(4, 5)$? **1**

(A) $\left(\dfrac{(2)(-5)+(-3)(4)}{-3+2}, \dfrac{(2)(2)+(-3)(5)}{-3+2}\right)$

(B) $\left(\dfrac{(-3)(-5)+(2)(4)}{-3+2}, \dfrac{(-3)(2)+(2)(5)}{-3+2}\right)$

(C) $\left(\dfrac{(-2)(-5)+(-3)(4)}{-3+2}, \dfrac{(-2)(2)+(-3)(5)}{-3+2}\right)$

(D) $\left(\dfrac{(2)(2)+(-3)(5)}{-3+2}, \dfrac{(2)(-5)+(-3)(4)}{-3+2}\right)$

4. Which of the following is the derivative of xe^{2x}? **1**

(A) $e^{2x}(1+x)$ (B) $e^{2x}(1+2x)$ (C) $2x^2e^{2x}$ (D) $\frac{1}{2}xe^{2x}$

5. Which of the following represents the complete solutions for $-180° < \theta \leq 180°$ to the equation **1**

$$\cos^2\frac{\theta}{2} = \frac{1}{4}$$

(A) 60°, 120° (B) 120°, 240° (C) ±60°, ±120° (D) ±120°

6. What should $\displaystyle\int \cos^2\frac{1}{2}x\,dx$ be transformed into, in order to find its primitive? **1**

(A) $\displaystyle\int \frac{1}{2} - \frac{\cos x}{2}\,dx$

(B) $\displaystyle\int \frac{1}{2} + \frac{\cos 2x}{2}\,dx$

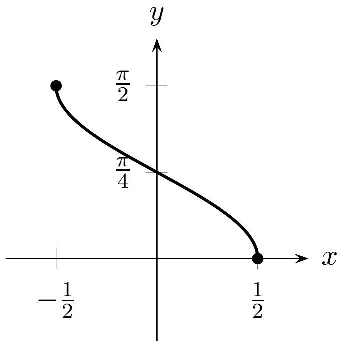
(C) $\displaystyle\int \frac{1}{2} - \frac{\cos 2x}{2}\,dx$

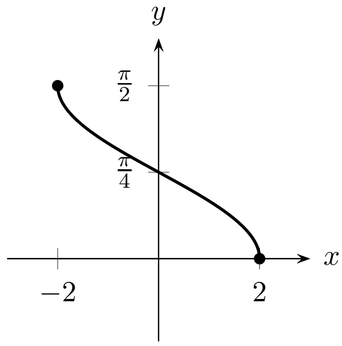
(D) $\displaystyle\int \frac{1}{2} + \frac{\cos x}{2}\,dx$

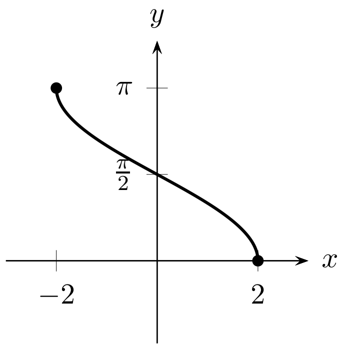
7. It is known that $\log_e x + \sin x = 0$ has a root close to $x = 0.5$. Using one application of Newton's method, which of the following gives a better approximation to 2 decimal places? **1**

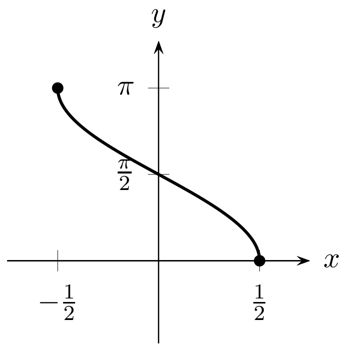
(A) 0.43 (B) 0.73 (C) 0.57 (D) 0.27

8. Which of the following graphs represents $y = \cos^{-1} 2x$? **1**

(A)


(C)


(B)


(D)


9. If $\sqrt{3}\cos x - \sin x \equiv R\cos(x+\alpha)$, which of the following gives the correct value of α? **1**

(A) $\dfrac{\pi}{6}$ (B) $\dfrac{5\pi}{6}$ (C) $\dfrac{7\pi}{6}$ (D) $\dfrac{11\pi}{6}$

10. Zac and Mitchell play a series of games. The series ends when one player has won two games. In any game the probability that Zac wins is $\frac{3}{5}$ and the probability that Mitchell wins is $\frac{2}{5}$. **1**

What is the probability that three games are played?

(A) $\dfrac{6}{25}$ (B) $\dfrac{19}{25}$ (C) $\dfrac{12}{25}$ (D) $\dfrac{18}{25}$

End of Section I.
Examination continues overleaf.

Section II: Short answer

Question 11 (15 Marks) Commence a NEW page. **Marks**

(a) Solve for x: $\dfrac{1}{x} > x$ **3**

(b) Evaluate $\displaystyle\int_0^2 \frac{dx}{\sqrt{16 - x^2}}$. **2**

(c) Find the exact value of $\sin\left(2\tan^{-1}\dfrac{3}{7}\right)$, showing full working. **3**

(d) Solve $\sin 2\theta = \sin\theta$, $0 \le \theta \le 2\pi$. **3**

(e) Using the substitution $u = \tan x$, find the exact value of **4**

$$\int_0^{\frac{\pi}{4}} \frac{\sec^2 x}{3 + \tan^2 x}\,dx$$

Question 12 (15 Marks) Commence a NEW page. **Marks**

(a) If $P(x) = x^3 - 6x^2 + ax - 4$, $a > 0$,

i. Given all the roots of $P(x) = 0$ are real and positive, and that one of the roots is the product of the other two roots, show that $a = 10$. **3**

ii. Show that $x - 2$ is a factor of $P(x) = x^3 - 6x^2 + 10x - 4$. **2**

(b) Air is being pumped into a spherical balloon at a rate of $20\,\text{cm}^3\text{s}^{-1}$. Find the rate of increase of the surface area of the balloon when the radius is $5\,\text{cm}$. **3**

(c) A particle moves along the x axis such that its velocity v ms^{-1} is given by

$$v^2 = -4x^2 + 8x + 32$$

i. By expressing the acceleration as a function in terms of x, prove that the particle is undergoing simple harmonic motion. **3**

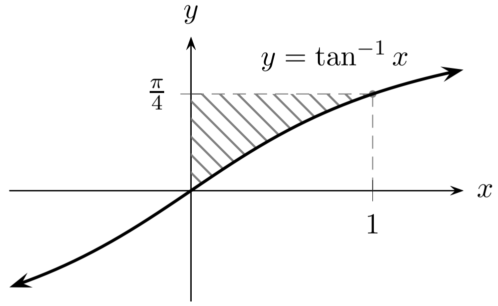
ii. Find the amplitude. **2**

iii. Find the maximum accceleration. **2**

Question 13 (15 Marks) Commence a NEW page. **Marks**

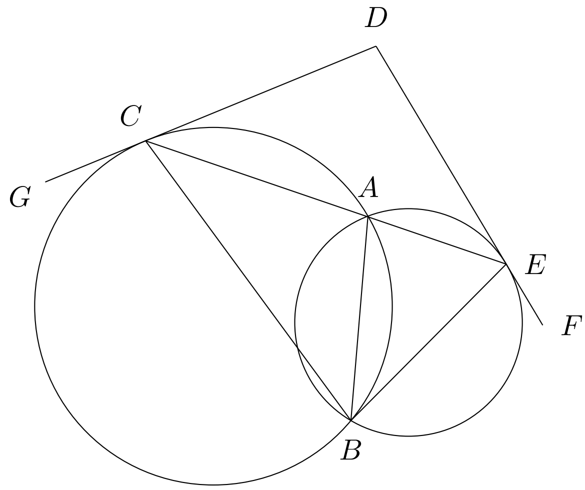
(a) Prove by mathematical induction that $5^n + 2 \times 11^n$ is divisible by 3, where n is a positive integer. **3**

(b) Show that the shaded area is $A = \frac{1}{2}\ln 2$ units^2. **3**



(c) Two circles intersect at A and B. CAE is a straight line where C is a point on the first circle and E is a point on the second circle. The tangents to the circles at C and E meet at D. **4**

Copy the diagram into your writing booklet.



Prove that $BCDE$ is a cyclic quadrilateral, without adding any construction lines.

(d) The acceleration of a raindrop which at time t seconds is falling with speed v metres per second is given by the equation

$$\frac{dv}{dt} = -\frac{1}{3}(v - 3g)$$

where g is a constant.

i. Show that $v = 3g + Ae^{-\frac{1}{3}t}$, where A is a constant, satisfies the above equation. **1**

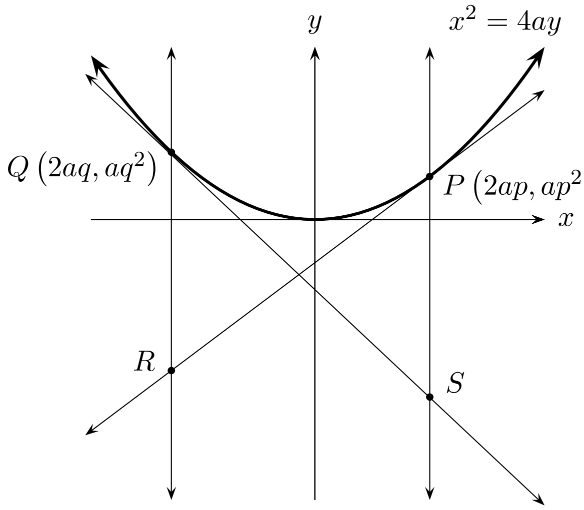
ii. Given that the initial velocity has a value of g, find the value of A. **1**

iii. After how many seconds is the raindrop falling with a speed of $2g$ metres per second? Give your answer correct to 1 decimal place. **2**

iv. What value does v approach as $t \to \infty$? **1**

Question 14 (15 Marks) Commence a NEW page. **Marks**

(a) $P(2ap, ap^2)$ and $Q(2aq, aq^2)$ are two points on the parabola $x^2 = 4ay$. The tangent at P and the line through Q parallel to the axis of the parabola meet at the point R.



The tangent at Q and the line through P parallel to the axis of the parabola meet at the point S.

i. Show that the equations of the tangents at P and Q are $y = px - ap^2$ and $y = qx - aq^2$ respectively. **2**

ii. Show that the coordinates of S and R are **2**

$$S\left(2ap, 2apq - aq^2\right) \qquad R\left(2aq, 2apq - ap^2\right)$$

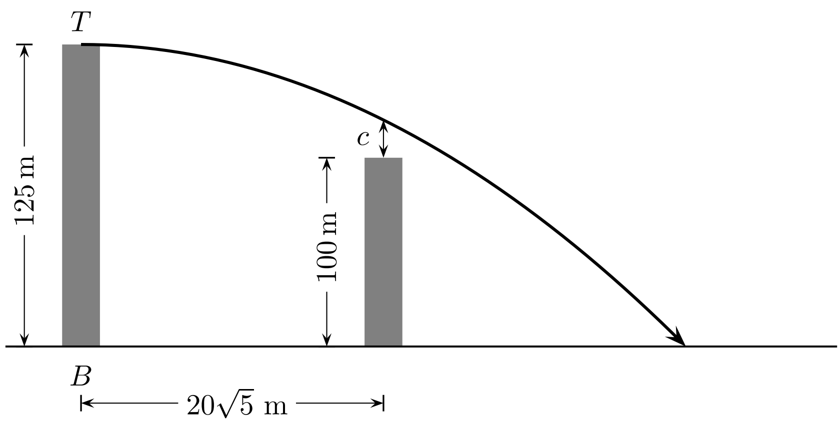
iii. Show that $PQRS$ is a parallelogram. **2**

iv. Show that the area of this parallelogram is $2a^2\left|p - q\right|^3$. **2**

Question 14 continues overleaf...

Question 14 continued from the previous page...

(b) A projectile is thrown horizontally from the top of a 125 m tower with velocity V metres per second. It clears a second tower of height 100 m by a distance of c metres, as shown. The two towers are $20\sqrt{5}$ metres apart.



i. The equations of motion for this system are **1**

$$\begin{cases} x = Vt \\ y = -5t^2 + 125 \end{cases}$$

(*Do not* prove this)

Where is the origin of the system being taken from?

ii. Show that $V = \dfrac{100}{\sqrt{25 - c}}$. **2**

iii. Prove that the minimum initial speed of the projectile to just clear the 100 m tower is $20\,\mathrm{ms}^{-1}$. **1**

iv. Hence, find how far past the 100 m tower will the projectile strike the ground. **2**

v. Determine the vertical component of the velocity of the projectile when it strikes the ground. **1**

End of paper.

STANDARD INTEGRALS

$$\int x^n\,dx = \frac{1}{n+1}x^{n+1}+C, \qquad n\neq -1;\quad x\neq 0 \text{ if } n<0$$

$$\int \frac{1}{x}\,dx = \ln x + C, \qquad x>0$$

$$\int e^{ax}\,dx = \frac{1}{a}e^{ax}+C, \qquad a\neq 0$$

$$\int \cos ax\,dx = \frac{1}{a}\sin ax + C, \qquad a\neq 0$$

$$\int \sin ax\,dx = -\frac{1}{a}\cos ax + C, \qquad a\neq 0$$

$$\int \sec^2 ax\,dx = \frac{1}{a}\tan ax + C, \qquad a\neq 0$$

$$\int \sec ax\tan ax\,dx = \frac{1}{a}\sec ax + C, \qquad a\neq 0$$

$$\int \frac{1}{a^2+x^2}\,dx = \frac{1}{a}\tan^{-1}\frac{x}{a}+C, \qquad a\neq 0$$

$$\int \frac{1}{\sqrt{a^2-x^2}}\,dx = \sin^{-1}\frac{x}{a}+C, \qquad a>0, -a<x<a$$

$$\int \frac{1}{\sqrt{x^2-a^2}}\,dx = \ln\left(x+\sqrt{x^2-a^2}\right)+C, \qquad x>a>0$$

$$\int \frac{1}{\sqrt{x^2+a^2}}\,dx = \ln\left(x+\sqrt{x^2+a^2}\right)+C$$

NOTE: $\ln x = \log_e x,\ x>0$

Answer sheet for Section I

Mark answers to Section I by fully blackening the correct circle, e.g "●"

STUDENT NUMBER: . . . .

Class (please ✔)

- ◯ 12M4A – Mr Weiss
- ◯ 12M4B – Mr Ireland
- ◯ 12M4C – Mr Fletcher
- ◯ 12M3C – Ms Ziaziaris
- ◯ 12M3D – Mr Lowe
- ◯ 12M3E – Mr Lam

1 – Ⓐ Ⓑ Ⓒ Ⓓ
2 – Ⓐ Ⓑ Ⓒ Ⓓ
3 – Ⓐ Ⓑ Ⓒ Ⓓ
4 – Ⓐ Ⓑ Ⓒ Ⓓ
5 – Ⓐ Ⓑ Ⓒ Ⓓ
6 – Ⓐ Ⓑ Ⓒ Ⓓ
7 – Ⓐ Ⓑ Ⓒ Ⓓ
8 – Ⓐ Ⓑ Ⓒ Ⓓ
9 – Ⓐ Ⓑ Ⓒ Ⓓ
10 – Ⓐ Ⓑ Ⓒ Ⓓ

Suggested Solutions

Section I

(Lowe)

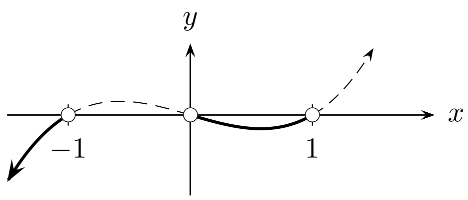
1. (C) **2.** (B) **3.** (A) **4.** (B) **5.** (D)
6. (D) **7.** (C) **8.** (D) **9.** (A) **10.** (C)

Question 11 (Lam)

(a) (3 marks)

- ✓ [1] for multiplying by square of denominator.
- ✓ [0] for entire part if only multiplying by denominator.
- ✓ [1] for each correct inequality.

$$\begin{aligned} \underset{\times x^2}{\frac{1}{x}} &> \underset{\times x^2}{x} \\ x &> x^3 \\ x^3 - x &< 0 \\ x(x^2-1) &< 0 \\ x(x-1)(x+1) &< 0 \end{aligned}$$



From the sketch,

$$x < -1 \text{ or } 0 < x < 1$$

(b) (2 marks)

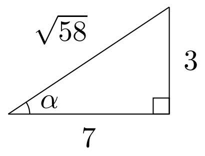
- ✓ [1] for correct primitive.
- ✓ [1] for correct evaluation of limits.

$$\begin{aligned} \int_0^2 \frac{dx}{\sqrt{16-x^2}} &= \left[\sin^{-1}\frac{x}{4}\right]_0^2 \\ &= \sin^{-1}\frac{1}{2} - \sin^{-1} 0 \\ &= \frac{\pi}{6} \end{aligned}$$

(c) (3 marks)

- ✓ [1] for drawing relevant right-angled triangle.
- ✓ [1] for expanding $\sin 2\alpha$.
- ✓ [1] for final answer.

Let $\alpha = \tan^{-1}\frac{3}{7}$. Then $\tan\alpha = \frac{3}{7}$:



$$\begin{aligned} \sin\left(2\tan^{-1}\frac{3}{7}\right) &\equiv \sin 2\alpha \\ &= 2\sin\alpha\cos\alpha \\ &= 2 \times \frac{3}{\sqrt{58}} \times \frac{7}{\sqrt{58}} \\ &= \frac{42}{58} = \frac{21}{29} \end{aligned}$$

(d) (3 marks)

- ✓ [1] for factorising expression into $\sin\theta(2\cos\theta - 1) = 0$.
- ✓ [1] for solutions in positive integral multiples of π.
- ✓ [1] for solutions in multiples of $\frac{\pi}{3}$.

$$\sin 2\theta = \sin\theta$$
$$2\sin\theta\cos\theta - \sin\theta = 0$$
$$\sin\theta(2\cos\theta - 1) = 0$$
$$\sin\theta = 0 \qquad \cos\theta = \frac{1}{2}$$
$$\theta = 0, \pi, 2\pi \qquad \theta = \frac{\pi}{3}, \frac{5\pi}{3}$$
$$\therefore \theta = 0, \pi, 2\pi, \frac{\pi}{3}, \frac{5\pi}{3}$$

(e) (4 marks)

✓ [1] for changing limits.
✓ [1] for making algebraic substitution.
✓ [1] for correct primitive.
✓ [1] for final answer.

$$\int_0^{\frac{\pi}{4}} \frac{\sec^2 x}{3+\tan^2 x}\,dx$$

Letting $u = \tan x$,

$$\begin{aligned}
\frac{du}{dx} &= \sec^2 x \\
\therefore du &= \sec^2 x\,dx \\
x = 0 &\quad \Rightarrow u = 0 \\
x = \frac{\pi}{4} &\quad \Rightarrow u = 1
\end{aligned}$$

$$\begin{aligned}
&\int_0^{\frac{\pi}{4}} \frac{\sec^2 x}{3+\tan^2 x}\,dx \\
&= \int_{u=0}^{u=1} \frac{\sec^2 x\,dx}{3+u^2} \\
&= \int_0^1 \frac{du}{3+u^2} \\
&= \frac{1}{\sqrt{3}}\left[\tan^{-1}\frac{u}{\sqrt{3}}\right]_0^1 \\
&= \frac{1}{\sqrt{3}}\left(\tan^{-1}\frac{1}{\sqrt{3}} - \tan^{-1} 0\right) \\
&= \frac{1}{\sqrt{3}} \times \frac{\pi}{6} \\
&= \frac{\pi}{6\sqrt{3}} \left(= \frac{\pi\sqrt{3}}{18}\right)
\end{aligned}$$

Question 12 (Lowe)

(a) i. (3 marks)

✓ [1] for $\alpha\beta = 2$.
✓ [1] for $\alpha + \beta = 4$.
✓ [1] for final answer.

$P(x) = x^3 - 6x^2 + ax - 4$. Let the roots be α, β and $\alpha\beta$.

- Sum of roots:

$$\alpha + \beta + \alpha\beta = -\frac{b}{a} = 6 \quad (12.1)$$

- Pairs of roots:

$$\alpha\beta + \alpha^2\beta + \beta^2\alpha = \frac{c}{a} = a \quad (12.2)$$

- Product of roots:

$$\begin{aligned}
\alpha\beta\,(\alpha\beta) &= -\frac{d}{a} = 4 \\
\alpha^2\beta^2 &= 4 \\
\therefore \alpha\beta &= 2 \qquad (12.3)
\end{aligned}$$

as roots are positive.

Substitute (12.3) into (12.1):

$$\begin{aligned}
\alpha + \beta + 2 &= 6 \\
\therefore \alpha + \beta &= 4 \qquad (12.4)
\end{aligned}$$

Substitute (12.4) into (12.2) to find a:

$$\begin{aligned}
\alpha\beta + \alpha\beta(\alpha+\beta) &= a \qquad (12.5) \\
2 + 2(4) &= a \\
\therefore a &= 10
\end{aligned}$$

ii. (2 marks)

✓ [2] for correct application of factor theorem.

If $x - 2$ is a factor then $P(2) = 0$.

$$\begin{aligned}
P(2) &= 2^3 - 6\left(2^2\right) + 10(2) - 4 \\
&= 8 - 24 + 20 - 4 = 0
\end{aligned}$$

(b) (3 marks)

✓ [1] for $\frac{dr}{dt}$.
✓ [1] for $\frac{dA}{dt} = \frac{dA}{dr} \times \frac{dr}{dt}$.
✓ [1] for final answer.

$$\begin{aligned}
\frac{dV}{dt} &= 20 = \frac{dV}{dr} \times \frac{dr}{dt} \\
V &= \frac{4}{3}\pi r^3 \\
\frac{dV}{dr} &= \frac{4}{3} \times \pi \times 3r^2 = 4\pi r^2 \\
\therefore \frac{dV}{dt} &= 20 = 4\pi r^2\Big|_{r=5} \times \frac{dr}{dt} \\
&= 4\pi \times 25 \times \frac{dr}{dt} \\
\therefore \frac{dr}{dt} &= \frac{1}{5\pi}
\end{aligned}$$

Now $\dfrac{dA}{dt} = \dfrac{dA}{dr} \times \dfrac{dr}{dt}$

$$\begin{aligned} SA &= 4\pi r^2 \\ \therefore \frac{dA}{dr} &= 8\pi r \\ \frac{dA}{dt} &= 8\pi r\Big|_{r=5} \times \frac{1}{5\pi} \\ &= 8\,\text{cm}^2\text{s}^{-1} \end{aligned}$$

(c) i. (3 marks)

- ✓ [1] for using $\frac{d}{dx}\left(\frac{1}{2}v^2\right)$ to find acceleration.
- ✓ [1] obtaining $\ddot{x} = -4x + 4$.
- ✓ [1] factorising and noting form $\frac{dv}{dt} = -n^2\left(x - x_0\right)$ for SHM.

$$\begin{aligned} v^2 &= -4x^2 + 8x + 32 \\ \frac{1}{2}v^2 &= -2x^2 + 4x + 16 \\ \ddot{x} = \frac{dv}{dt} &= \frac{d}{dx}\left(\frac{1}{2}v^2\right) \\ &= \frac{d}{dx}\left(-2x^2 + 4x + 16\right) \\ &= -4x + 4 = -4(x-1) \end{aligned}$$

As acceleration is proportion to the opposite direction of displacement, hence the particle is moving in simple harmonic motion with centre at $x = 1$.

ii. (2 marks)

- ✓ [1] for $x = 4$, $x = -2$.
- ✓ [1] for finding amplitude.

The amplitude occurs when $\dot{x} = 0$.

$$\begin{aligned} -4x^2 + 8x + 32 &= 0 \\ x^2 - 2x - 8 &= 0 \\ (x-4)(x+2) &= 0 \\ \therefore x &= 4, -2 \\ a = \frac{4 + |-2|}{2} &= 3 \end{aligned}$$

As the centre of motion is $x = 1$ and particle's maximum displacement is 4 or -2, therefore the amplitude is $a = 3$.

iii. (1 mark)

Maximum acceleration occurs at the amplitude, i.e. $x = 4$ or $x = -2$.

$$\begin{aligned} \ddot{x} &= -4(x-1)\Big|_{x=-2} \\ &= -4(-2-1) = 12 \end{aligned}$$

Question 13 (Ziaziaris)

(a) (3 marks)

- ✓ [1] for proving base case.
- ✓ [1] for inductive step.
- ✓ [1] for required proof.

Let $P(n)$ be the statement $5^n + 2 \times 11^n$ is divisible by 3, i.e.

$$5^n + 2 \times 11^n = 3J$$

where $J \in \mathbb{N}$.

- Base case: $P(1)$:

  $$5^1 + 2 \times 11 = 5 + 22 = 27$$

  which is divisible by 3. Hence $P(1)$ is true.
- Inductive step:
  - Assume $P(k)$ is true for some $k \in \mathbb{N}$, $k < n$, i.e.

    $$5^k + 2 \times 11^k = 3M$$

    where $M \in \mathbb{N}$. Alternatively,

    $$5^k = 3M - 2 \times 11^k$$
  - Examine $P(k+1)$:

    $$\begin{aligned} &5^{k+1} + 2 \times 11^{k+1} \\ &= 5^k 5^1 + 2 \times 11^{k+1} \\ &= 5\left(3M - 2 \times 11^k\right) + 2 \times 11^{k+1} \\ &= 3 \times 5M - 10 \times 11^k + 2 \times 11 \times 11^k \\ &= 3 \times 5M - 10 \times 11^k + 22 \times 11^k \\ &= 3 \times 5M + 12 \times 11^k \\ &= 3\underbrace{\left(5M + 4 \times 11^k\right)}_{\in \mathbb{N}} \equiv 3P \end{aligned}$$

    where $P \in \mathbb{N}$. Hence $P(k+1)$ is true.

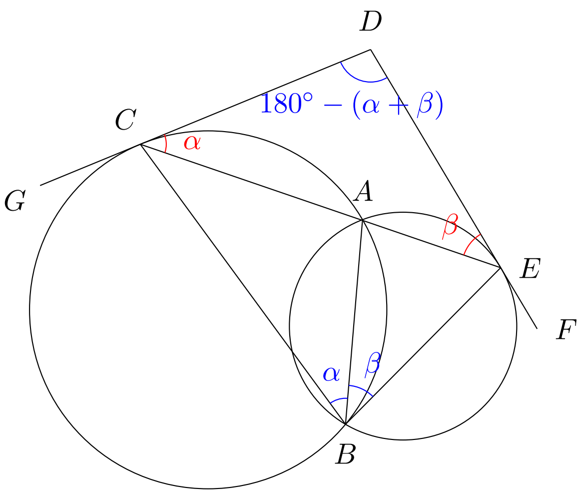
  Since $k \in \mathbb{N}$ and truth in $P(k)$ also leads to truth in $P(k+1)$, therefore $P(n)$ is true by induction.

(b) (3 marks)

- ✓ [1] for coverting integrand to $\frac{\sin y}{\cos y}$
- ✓ [1] for correct primitive
- ✓ [1] for final answer.

$$\begin{aligned} A &= \int_0^{\frac{\pi}{4}} \tan y\, dy \\ &= \int_0^{\frac{\pi}{4}} \frac{\sin y}{\cos y}\, dy \\ &= \Big[-\log_e(\cos y)\Big]_0^{\frac{\pi}{4}} \\ &= -\log_e \cos\frac{\pi}{4} + \log_e \cos 0 \\ &= -\log_e \frac{1}{\sqrt{2}} + \log_e 1 \\ &= -\log_e 2^{-\frac{1}{2}} = \frac{1}{2}\log_e 2 \end{aligned}$$

(c) (4 marks) – marking scheme embedded inline. Presence of ✓indicates 1 mark.



- ✓ Let $\angle DCE = \alpha$ and $\angle DEC = \beta$. $\therefore \angle CBA = \alpha$ ($\angle$ in alternate segment)
- ✓ Similarly, $\angle ABE = \beta$ ($\angle$ in alternate segment)
- ✓ Also, $\angle CDE = 180^\circ - (\alpha + \beta)$. (Angle sum of $\triangle CDE$)
- • Hence $\angle CDE + \angle CBE = 180^\circ$
- ✓ Opposite $\angle$ in $BCDE$ are supplementary. Hence $BCDE$ is a cyclic quadrilateral.

(d) i. (1 mark)

$$\begin{aligned} \underset{-3g}{v} &= \underset{-3g}{3g} + Ae^{-\frac{1}{3}t} \\ v - 3g &= Ae^{-\frac{1}{3}t} \\ \frac{dv}{dt} &= -\frac{1}{3}\underbrace{Ae^{-\frac{1}{3}t}}_{=(v-3g)} \\ &= -\frac{1}{3}(v-3g) \end{aligned}$$

ii. (1 mark)

$$\begin{aligned} t = 0&, v = g \\ \therefore g &= 3g + Ae^0 \\ \therefore A &= -2g \end{aligned}$$

iii. (2 marks)

$$\begin{aligned} v = 2g&, t = ? \\ 2g &= 3g - 2ge^{-\frac{1}{3}t} \\ -g &= -2ge^{-\frac{1}{3}t} \\ \frac{1}{2} &= e^{-\frac{1}{3}t} \\ -\frac{1}{3}t &= \log_e \frac{1}{2} = -\log_e 2 \\ \therefore t &= 3\log_e 2 \approx 2.1\,\text{seconds} \end{aligned}$$

iv. (1 mark)
As $t \to \infty$, $v \to 3g$.

Question 14 (Ireland/Fletcher)

(a) i. (2 marks)

✓ [1] for proving $\frac{dy}{dx} = p$ at P.

✓ [1] for equation of tangent at P.

$$x^2 = 4ay \quad \Rightarrow \quad y = \frac{x^2}{4a}$$
$$\frac{dy}{dx} = \frac{2x}{4a} = \frac{x}{2a}$$

At $x = 2ap$,

$$\frac{dy}{dx} = \frac{2ap}{2a} = p$$

Equation of the tangent at P:

$$y - ap^2 = p(x - 2ap) = px - 2ap^2$$
$$y = px - ap^2$$

Similarly, the tangent at Q is

$$y = qx - aq^2$$

ii. (2 marks)

✓ [1] for each y coordinate.

Coordinates of S arise from the intersection of $x = 2ap$ and $y = qx - aq^2$:

$$y = q(2ap) - aq^2 = 2apq - aq^2$$
$$\therefore S\left(2ap, 2apq - aq^2\right)$$

Coordinates of R arise from the intersection of $x = 2aq$ and $y = px - ap^2$:

$$y = p(2aq) - ap^2 = 2apq - ap^2$$
$$\therefore R\left(2aq, 2apq - ap^2\right)$$

iii. (2 marks)

✓ [1] for showing $PS \parallel QR$.

✓ [1] for showing $PS = QR$.

$$d_{PS} = \sqrt{\cancel{(2ap - 2ap)^2} + (ap^2 - (2apq - aq^2))^2}$$
$$= \sqrt{(a(p-q)^2)^2}$$
$$= a(p-q)$$
$$d_{QR} = \sqrt{\cancel{(2aq - 2aq)^2} + (aq^2 - (2apq - ap^2))}$$
$$= \sqrt{a(q-p)^2} = \sqrt{a(p-q)^2}$$
$$= a(p-q)$$

As $PS = QR$, hence one pair of opposite sides equal and parallel. Hence $PQRS$ is a parallelogram.

Alternatively, if $PQRS$ is a parallelogram, then the diagonals bisect each other; i.e. QS and PR share the same midpoint. Show via midpoint formula results in

$$MP_{QS} = \left(\frac{a(p+q)}{2}, apq\right)$$
$$MP_{PR} = \left(\frac{a(p+q)}{2}, apq\right)$$

iv. (2 marks)

✓ [1] for h (fully)

✓ [1] for area.

- Use $A = bh$.
- h is perpendicular distance from Q to PS.
- Use $b = d_{PS}$.

Using the perpendicular dist formula with $x = 2ap$ & $Q(2aq, aq^2)$:

$$h = \frac{|ax_1 + by_1 + c|}{\sqrt{a^2 + b^2}}$$
$$= \frac{|2aq(1) + 0 - 2ap|}{\sqrt{1^2 + 0}}$$
$$= \frac{|2aq - 2ap|}{1} = 2a\,|q - p|$$
$$= 2a\,|p - q|$$

As $|p - q| = |q - p|$.

$$A = bh$$
$$= 2a\,|p - q| \times a(p-q)^2$$
$$= 2a^2\,|p - q|^3$$

(b) i. (1 mark)
Origin is at the base of tower.

ii. (2 marks)
✓ [1] for $100 + c = -5\left(\frac{400\times 5}{V^2}\right) + 125$.
✓ [1] for final result shown.
When $x = 20\sqrt{5}$, $y = 100 + c$. Using $x = Vt$,

$$\begin{aligned} 20\sqrt{5} &= Vt \\ \therefore t &= \frac{20\sqrt{5}}{V} \end{aligned}$$

Substitute into $y = -5t^2 + 125$,

$$\begin{aligned} 100 + c &= -5\left(\frac{20\sqrt{5}}{V}\right)^2 + 125 \\ &= -5\left(\frac{400\times 5}{V^2}\right) + 125 \\ -25 + c &= -5 \times \frac{400 \times 5}{V^2} \\ 25 - c &= \frac{25 \times 400}{V^2} \\ V^2 &= \frac{10\,000}{25 - c} \\ \therefore V &= \frac{100}{\sqrt{25 - c}} \quad (V > 0) \end{aligned}$$

iii. (1 mark) Projectile just clears tower when $c = 0$.

$$V = \left.\frac{100}{\sqrt{25 - c}}\right|_{c=0} = 20\,\text{ms}^{-1}$$

iv. (2 marks)
✓ [1] for $x = 100$.
✓ [1] for final answer.
Projectile strikes ground when $y = 0$.

$$\begin{aligned} -5t^2 + 125 &= 0 \\ 5t^2 &= 125 \\ \therefore t^2 &= 25 \quad \Rightarrow \quad t = 5 \end{aligned}$$

When $t = 5$,

$$x = Vt = 20 \times 5 = 100$$

Hence projectile will strike the ground $100 - 20\sqrt{5}$ metres past the second tower.

v. (1 mark)

$$\begin{aligned} y &= -5t^2 + 125 \\ \dot{y} &= -10t\Big|_{t=5} = -50\,\text{ms}^{-1} \end{aligned}$$