NORTH SYDNEY BOYS HIGH SCHOOL

2004
TRIAL HIGHER SCHOOL CERTIFICATE EXAMINATION

Mathematics

Extension 2

Examiner: G. Rezcallah

General Instructions

- Reading time – 5 minutes
- Working time – 3 hours
- Write on one side of the paper (with lines) in the booklet provided
- Write using blue or black pen
- Board approved calculators may be used
- All necessary working should be shown in every question
- Each new question is to be started on a **new page.**

- Attempt all questions

Class Teacher:
(Please tick or highlight)
- O Mr Ee
- O Mr Rezcallah
- O Mr Barrett

Student Number:____________________

(To be used by the exam markers only.)

Question No	1	2	3	4	5	6	7	8	Total	Total
Mark	/15	/15	/15	/15	/15	/15	/15	/15	/120	/100

QUESTION 1 Begin a new page.

Marks

(a) Find: (i) $\int \frac{x^4}{\sqrt{x^5 - 7}} dx$. 2

(ii) $\int \frac{1}{e^x + e^{-x}} dx$ 2

(b) Evaluate $\int_2^6 x\sqrt{6-x}\,dx$ Using the substitution $u^2 = 6 - x$. 3

(c) (i) Find the constants A and B such that

$$\frac{1}{\cos x} = \frac{A\cos x}{1-\sin x} + \frac{B\cos x}{1+\sin x}$$ 2

(ii) Hence, find the exact value of the integral $\int_0^{\frac{\pi}{6}} \sec x\,dx$ 2

(d) Show that $\int_0^a f(x)dx = \int_0^a f(a-x)dx$ 2

Hence, evaluate the integral $\int_0^{\frac{\pi}{2}} \frac{e^{\sin x}}{e^{\sin x} + e^{\cos x}} dx$. 2

QUESTION 2 Begin a new page.

Marks

(a) Consider the complex numbers $Z_1 = \sqrt{2}\,(1 + i\sqrt{3})$ and $Z_2 = 2\sqrt{6}\,(1 + i)$

(i) Express $z = \dfrac{Z_1}{Z_2}$ exactly in the form $x + iy$, where x and y are real. 2

(ii) Write Z_1 , Z_2 and z in modulus/argument form. 2

(iii) Hence, find the exact value of $\cos\dfrac{\pi}{12}$. 1

(iv) On an Argand diagram draw the vectors $\overrightarrow{OA}, \overrightarrow{OB}$, and $\overrightarrow{OS}$, to represent Z_1 , Z_2 and $Z_1 - Z_2$ respectively. 2

(b) Indicate on an Argand diagram the region which contains the point P representing z when :

(i) $\text{Re}(z + iz) \geq 2$ 2

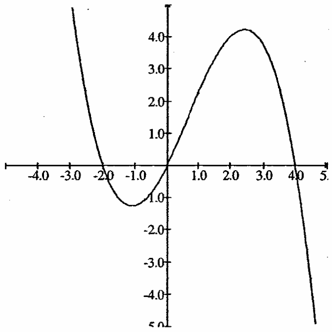
(ii) $1 \leq |z - 1 - i| \leq 3$ where $z = x + iy$ 2

(c) By applying De Moivre's theorem and by also expanding $(\cos\theta + i\sin\theta)^5$, express $\sin 5\theta$ as a polynomial in $\sin\theta$. 4

QUESTION 3 Begin a new page.

Marks

(a) The diagram shows the graph of $y = f(x)$ which passes through the origin and cuts the x axis at $x = -2$ and $x = 4$. The point $(1\ ,\ 2\frac{1}{4})$ belongs to the curve.



(i) Write down the equation of $y = f(x)$. 2

On separate diagrams, sketch each of the following: 10

(ii) $y = -f(x)$

(iii) $y = f(-x)$

(iv) $y^2 = f(x)$

(v) $y = |f(|x|)|$

(vi) $y = \dfrac{1}{1 - f(x)}$

(b) Consider in the set of complex numbers C:

w the cubic root of unity, $x = a + b$, $y = aw + bw^2$ and $z = aw^2 + bw$.

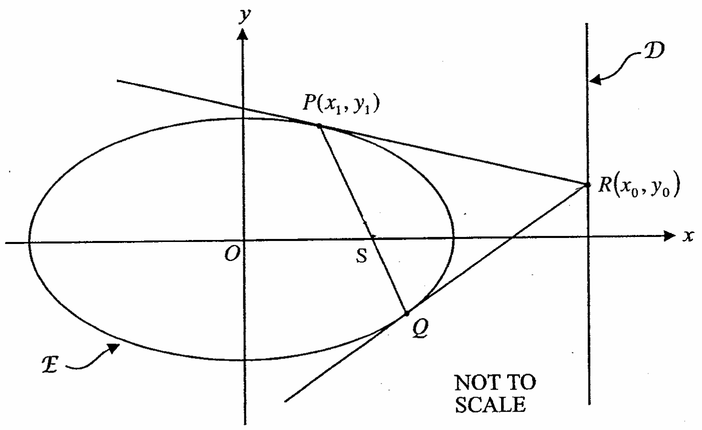
(i) Show that $1 + w + w^2 = 0$ 1

(ii) Prove that $x^2 + y^2 + z^2 = 6ab$ 2

QUESTION 4 Begin a new page.

Marks

(a)



The ellipse E with equation $\frac{x^2}{16} + \frac{y^2}{9} = 1$ has a directrix D as shown in the diagram.
Point $R(x_0, y_0)$ lies on D.
PQ is the chord of contact from R where P is the point (x_1, y_1).

(i) Write down the equation of D and the coordinates of the focus S. 2

(ii) Show that the tangent at a point P (x_1, y_1) has an equation of

$$\frac{xx_1}{16} + \frac{yy_1}{9} = 1.$$ 2

(iii) Write down the equation of chord PQ and show that the focus S lies on PQ. 2

(iv) Show that the angle subtended by PR at the focus S is 90°. 3

(v) Hence, deduce that the points P, S, and R are concyclic.. 1

QUESTION 4 (Continued)

Marks

(b)

(i) Sketch the graph of $f(x)=\sqrt{(x+1)^2}+\sqrt{(x-1)^2}$. 2

(ii) Hence, solve the equation $-2 \le \sqrt{(x+1)^2}+\sqrt{(x-1)^2} < 2$ 1

(c) The equation $x^3+kx+r=0$ has roots $\alpha, \beta,$ and γ.

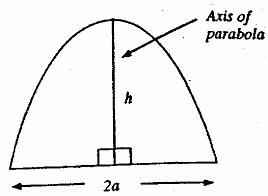
Find the value of the expression $\alpha^3+\beta^3+\gamma^3$ 2

QUESTION 5 **Begin a new page.**

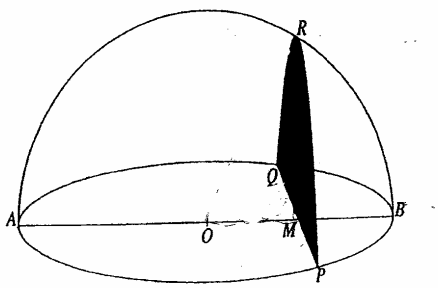
Marks

(a) (i) A parabolic segment has height h and width 2a.

Use Simpson's rule with three function values to show that the exact area of this segment is $\frac{4}{3}ah$. 2



In the diagram below, a tent has a circular base with centre O and radius a, and AOB is a diameter of the base. - The shaded area PMQR is a typical cross section of the tent perpendicular to AB, and meets AB at a point M distant x from O. The curve PRQ is a parabola with axis RM and QM = RM.



(ii) Use part (i) to show that the shaded area PMQR *is* $\frac{4}{3}(a^2 - x^2)$. 2

(iii) Find the volume of the tent. 2

(b) Factorise the polynomial $P(z) = z^4 - 2z^2 + 8z - 3$ fully over C
Given that $P(1-\sqrt{2}i) = 0$. 4

(c) (i) By considering the perfect square $(\sqrt{x} - \sqrt{y})^2$ where x and y are positive, prove that $\frac{x+y}{2} \geqslant \sqrt{xy}$. 2

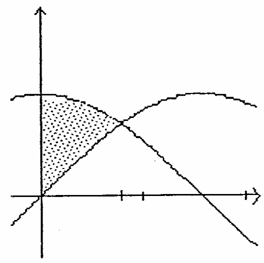
(ii) Hence, if a, b, c and d are positive numbers, prove that :

$$4(ab + bc + cd + da) \leq (a + b + c + d)^2$$ 3

QUESTION 6 Begin a new page.

Marks

(a) In the diagram, the shaded region is bounded by the y axis and the curves $y = \cos x$ and $y = \sin x$.



(i) Show that the curves intersect at $\left(\frac{\pi}{4}, \frac{1}{\sqrt{2}}\right)$ 1

(ii) The shaded region is rotated about the y axis.
Find the exact value of the volume obtained by this rotation, using the method of cylindrical shells. 5

(b) By considering the binomial expansion of $(1+i)^n$
Show that $1 - \binom{n}{2} + \binom{n}{4} - \binom{n}{6} + \ldots\ldots = 2^{\frac{n}{2}} \cos \frac{n\pi}{4}$ 2

(c) The normal at a point $P\left(x_1, \frac{9}{x_1}\right)$ on the rectangular hyperbola $xy = 9$ meets the curve again at another point A.

(i) Prove that the equation of intersection of the normal at P and the rectangular hyperbola is $x_1^3 x^2 + (81 - x_1^4)x - 81x_1 = 0$ 2

(ii) Hence, prove that the coordinates of A are $\left(-\frac{81}{x_1^3}, -\frac{x_1^3}{9}\right)$ 2

(iii) Let M be the midpoint of AP. Derive the cartesian equation of the locus of M. 3

QUESTION 7 Begin a new page.

Marks

(a) Consider the word **SOCCER**.

How many:

(i) six-letter different arrangements 1

(ii) selections of 4 letters 2

can be made from the letters in the word **SOCCER**?

(b) A particle of mass m is projected vertically upward under gravity in a medium in which the resistance is proportional to square of the velocity (mkv^2), where k is a constant.

(i) Show that the terminal speed V in the medium is $\sqrt{\dfrac{g}{k}}$ 1

If the speed of projection is equal to the terminal velocity V in the medium, show that:

(ii) the particle reaches a maximum height of $\dfrac{V^2}{2g}\ln 2$ above the point of projection. 3

(iii) the time taken to reach its maximum height is $\dfrac{\pi V}{4g}$ 4

(iv) the time t in the downward motion as a function of v is given by

$$t = \frac{1}{2\sqrt{gk}}\ln\left(\frac{V+v}{V-v}\right)$$ 4

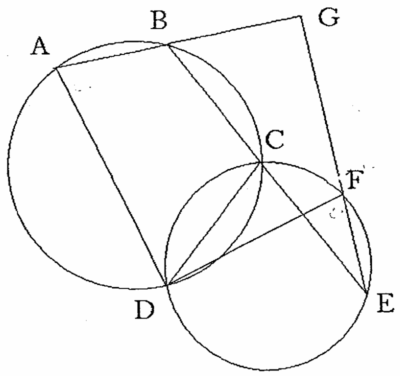
QUESTION 8 Begin a new page.

Marks

(a) If $I_n = \int_0^1 (1-x^2)^n \, dx$ for $n \geq 0$, show that $I_n = \frac{2n}{1+2n} I_{n-1}$ for $n \geq 1$. 3

Hence, find an expression for I_n in terms of n for $n \geq 1$. 2

(b)



Two circles intersect at C and D. ABCD is a cyclic quadrilateral in one circle.

BC produced meets the other circle at E. C, F, E and D are concyclic points.

AB produced meets EF produced at G.

Prove that GFDA is a cyclic quadrilateral. 4

(c) A sequence is defined by the recurrence relationship:

$U_1 = 1$ and $U_{n+1} = \frac{1}{2}\left[U_n + \frac{2}{U_n}\right]$ when $n \geq 1$, n a positive integer

(i) Prove by mathematical induction: $\frac{U_n - \sqrt{2}}{U_n + \sqrt{2}} = \left(\frac{1-\sqrt{2}}{1+\sqrt{2}}\right)^{2^{n-1}}$ 4

(ii) Hence, show that for n sufficiently large, U_n is very close to $\sqrt{2}$ 2