NORTH SYDNEY GIRLS HIGH SCHOOL

2013

TRIAL HSC EXAMINATION

Mathematics

General Instructions

- Reading Time – 5 minutes
- Working Time – 3 hours
- Write using black or blue pen.
- Board-approved calculators may be used
- A table of standard integrals is provided at the back of this paper
- Show all necessary working in Questions 11–16

Total Marks – 100

Section I **10 Marks**

- Attempt Questions 1–10
- Allow about 15 minutes for this section.

Section II **90 Marks**

- Attempt Questions 11–16
- Allow about 2 hours 45 minutes for this section.

Student Number: ____________________________ **Teacher:** ___________________________

Student Name: ______________________________

QUESTION	MARK
1 – 10	/10
11	/15
12	/15
13	/15
14	/15
15	/15
16	/15
TOTAL	/100

Section I

10 marks
Attempt Questions 1–10
Allow about 15 minutes for this section

Use the multiple-choice answer sheet for Questions 1–10.

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1 Which of the following represents $\log_e\left(\dfrac{10^3}{e^{10}-10}\right)$, evaluated to four significant figures?

(A) –1.3427
(B) –1.343
(C) –3.0918
(D) –3.092

2 What is the equation of the directrix of the parabola $x^2 = -8y$?

(A) $x = 2$
(B) $y = 2$
(C) $x = 8$
(D) $y = -8$

3 What is $8^3 \times 6^{\frac{1}{2}} \div 32^{\frac{3}{2}}$ in simplest form?

(A) $4\sqrt{3}$
(B) $2\sqrt{3}$
(C) $3\sqrt{2}$
(D) $4\sqrt{2}$

4 What is the sum of the first 12 terms of the following arithmetic series?

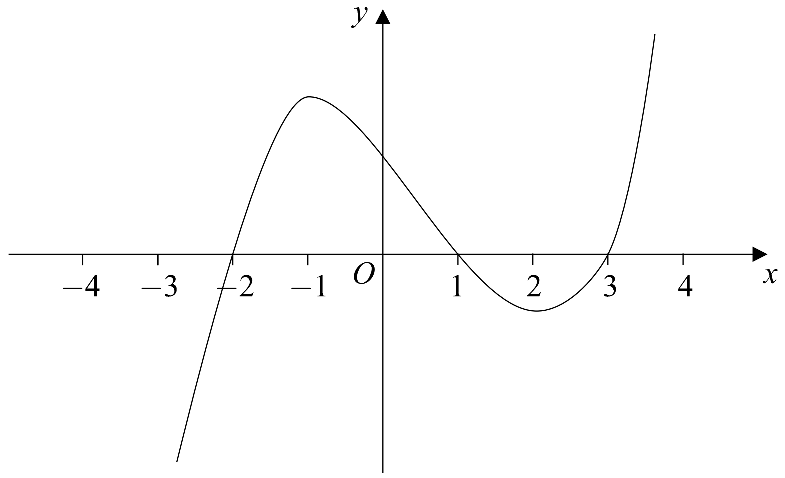
$$-20-13-6+1+...$$

(A) 222
(B) −702
(C) 264
(D) −744

5 The quadratic equation $5x^2 - 7x + 1 = 0$ has roots α and β.
What is the value of $\dfrac{3}{\alpha}+\dfrac{3}{\beta}$?

(A) −21
(B) 7
(C) 21
(D) −7

6 The graph of $y = f\left(x\right)$ is drawn below. It has a maximum turning point at (–1, 10).



What are the coordinates of the maximum point of the curve $y = \frac{1}{2} f\left(x+1\right)$?

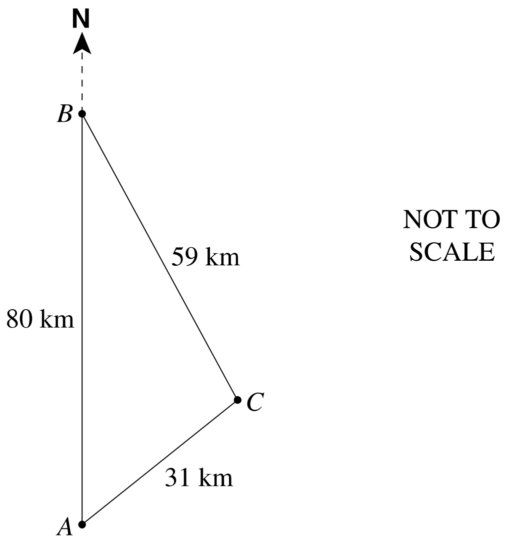
(A) $\left(0, 5\right)$

(B) $\left(-\frac{1}{2}, 5\right)$

(C) (–1, 5)

(D) (–2, 5)

7 In the diagram below, Town *B* is 80 km due north of town *A* and 59 km from Town *C*. Town *A* is 31 km from Town *C*.



What is the bearing of Town *C* from Town *B*?

(A) 019°

(B) 122°

(C) 161°

(D) 341°

8 If $y = 3\cos^4 x$, what is $\frac{dy}{dx}$?

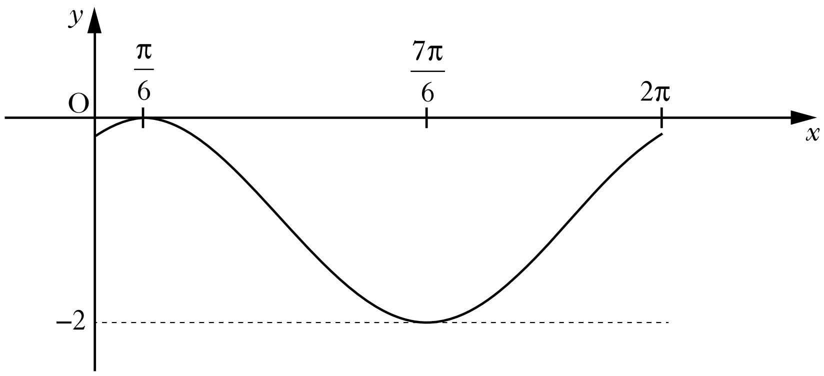
(A) $12\cos^3 x \sin x$

(B) $12\cos^3 x$

(C) $-12\cos^3 x \sin x$

(D) $-12\sin^3 x$

9 What is the equation of this curve?



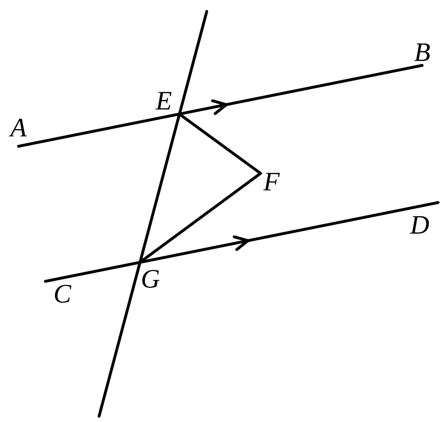
(A) $y = \cos\left(x - \frac{\pi}{6}\right) - 1$

(B) $y = \cos\left(x - \frac{\pi}{6}\right) + 1$

(C) $y = \cos\left(x + \frac{\pi}{6}\right) - 1$

(D) $y = \cos\left(x + \frac{\pi}{6}\right) + 1$

10 In the diagram below, $AB \parallel CD$, EF bisects $\angle BEG$ and GF bisects $\angle EGD$.
What is the size of $\angle EFG$?



(A) 30°

(B) 45°

(C) 60°

(D) 90°

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Section II

90 marks
Attempt Questions 11–16
Allow about 2 hours and 45 minutes for this section

Answer each question in a NEW writing booklet. Extra pages are available

In Questions 11–16, your responses should include relevant mathematical reasoning and/or calculations.

Question 11 (15 Marks) Start a NEW Writing Booklet

(a) Express $\dfrac{2}{2-\sqrt{3}}$ in the form $m+n\sqrt{3}$, where m and n are integers. **1**

(b) Solve $x-2=\sqrt{3x-2}$ **3**

(c) The first and fourth terms of a geometric series are 256 and 2048 respectively.

(i) What is the value of the common ratio? **1**

(ii) Given that the sum of the first n terms is 261 888, find the value of n. **2**

(d) Find $\dfrac{dy}{dx}$ when

(i) $y=\left(4x^2+3x+2\right)^{10}$. **2**

(ii) $y=x^2\tan x$. **2**

Question 11 continues on page 7

Question 11 (continued)

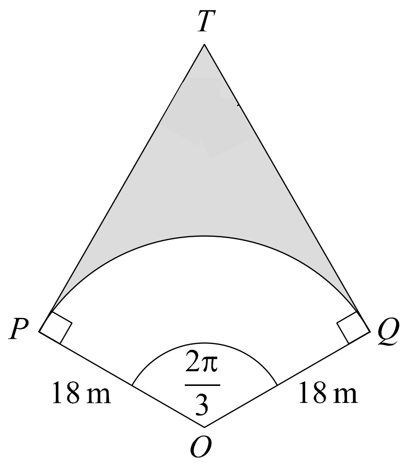
(e) The diagram below shows a sector OPQ of a circle with centre O.

The radius of the circle is 18 m and $\angle POQ = \frac{2\pi}{3}$.

It also shows the tangents at the points P and Q intersecting at T.

$\Delta POT \equiv \Delta QOT$ (Do NOT prove)

$\angle OPT = \angle OQT = \frac{\pi}{2}$.



(i) Find the area of sector POQ. **1**

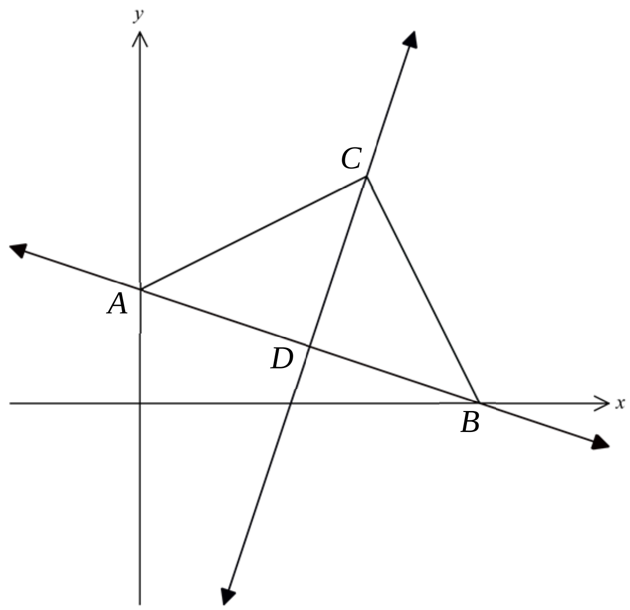
(ii) Show that $PT = 18\sqrt{3}$ m. **1**

(iii) Find the area of the shaded region. **2**
Leave your answer correct to 3 significant figures.

End of Question 11

Question 12 (15 Marks) Start a NEW Writing Booklet

(a) The diagram below shows ΔABC and its vertices $A(0,2)$, $B(6,0)$ and $C(4,k)$.
The line CD is the perpendicular bisector of AB.



(i) Find the angle of inclination that the line AB makes. with the positive direction of the x-axis. Leave answer correct to the nearest minute. **2**

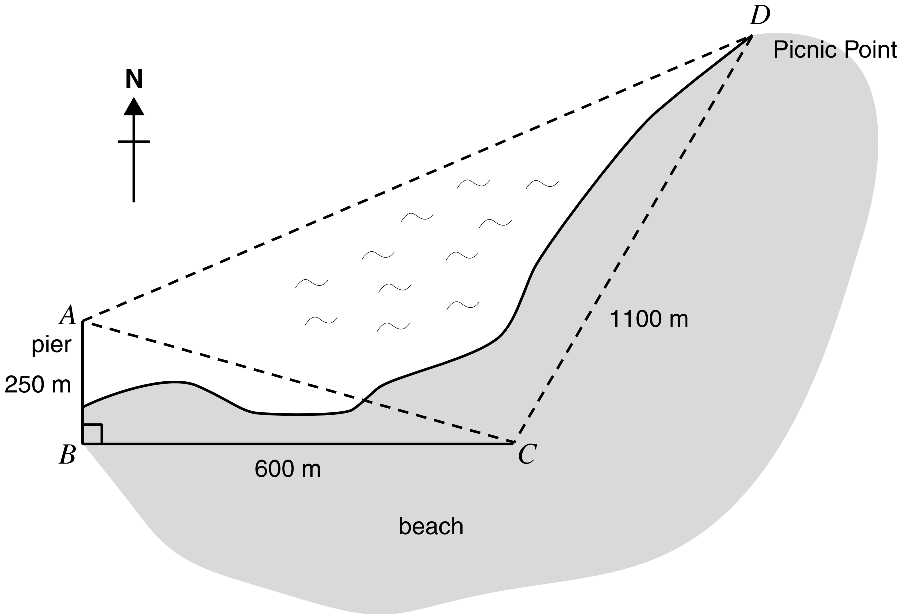
(ii) Show that the equation of CD is $3x - y - 8 = 0$. **2**

(iii) If $C(4,k)$, show that $k = 4$. **1**

(iv) If the line AB has the equation $x + 3y - 6 = 0$, find the area of ΔABC. **2**

Question 12 continues on page 9

(b) The diagram below shows the location of the Pier-to-Point swimming race. Swimmers enter the water at the northern end of the pier (A) and swim directly to Picnic Point (D)



Nilmot wants to find the distance competitors have to swim.
He measures the length AB of the pier and finds that it is 250 m.
He then starts at the southern end of the pier (B) and measures 600 m due east along the beach to C.

Using a compass, he finds that the bearing of D from C is $030°$.
He then measures the distance from C to D and finds it is 1100 m.

(i) Show that the distance, AC, is 650 m. **1**

(ii) Find the size of $\angle BCA$ to the nearest degree. **1**

(iii) Hence, find the distance, AD that the competitors have to swim. Leave your answer correct to the nearest metre. **2**

(c) (i) Find $\frac{d}{dx}\left(4x^3 - 6x + 1\right)$. **1**

(ii) Evaluate $\int_2^3 \frac{2x^2 - 1}{4x^3 - 6x + 1}\,dx$, leaving your answer in the form $p\log_e q$, where p and q are rational numbers. **3**

End of Question 12

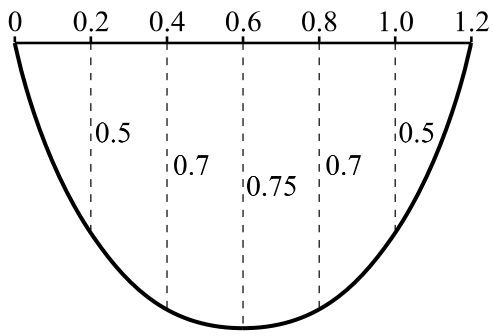
Question 13 (15 Marks) Start a NEW Writing Booklet

(a) Prove that $\sec^2\theta + \operatorname{cosec}^2\theta = \sec^2\theta\operatorname{cosec}^2\theta$. **2**

(b) (i) Expand $\left(\sqrt{3}u - 1\right)\left(u - \sqrt{3}\right)$. **1**

(ii) Hence solve $\sqrt{3}\tan^2\theta - 4\tan\theta + \sqrt{3} = 0$ for $0 \le \theta \le 2\pi$. **2**

(c) Farmer Rekrap digs ditches for flood relief. He experiments with different cross-sections. Assume that the surface of the ground is horizontal



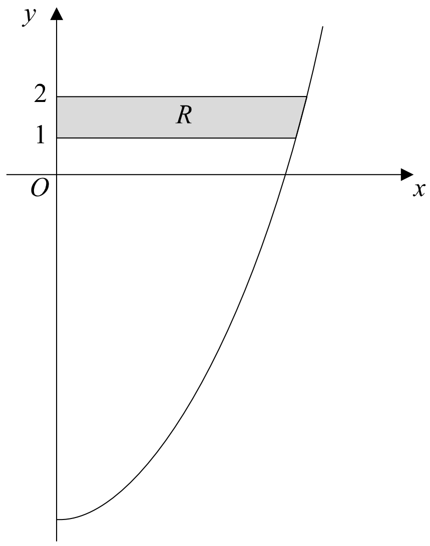
The diagram above shows the cross-section of one ditch, with measurements in metres. The width of the ditch is 1.2 m

By using the trapezoidal rule with 6 intervals to estimate the cross-sectional area, find the volume that can be contained in a 50-metre length of this ditch. **3**

Question 13 continues on page 11

Question 13 (continued)

(d) The diagram below shows the curve $y = x^2 - 9$ for $x \geq 0$.
The shaded region R is bounded by the curve, the lines $y = 1$ and $y = 2$, and the y-axis.



Find the volume of the solid of revolution when the region R is rotated about the y-axis. **2**

(e) A particle movies in a straight line.
At time t seconds, it has velocity v ms^{-1}, where $v = 6t^2 - 8e^{-4t} + 9$

(i) Find the particle's initial acceleration. **2**

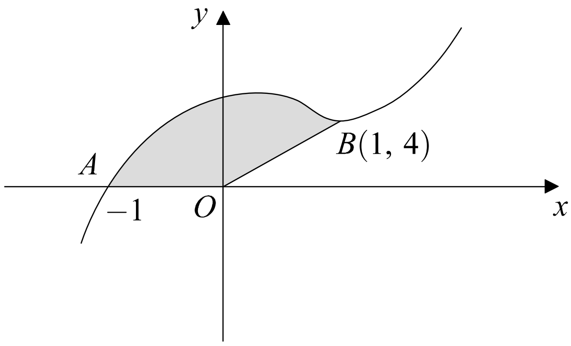
(ii) In what direction is the particle moving initially? **1**

(ii) Initially, the particle is at the origin.
Find an expression for the displacement of the particle at time t. **2**

End of Question 13

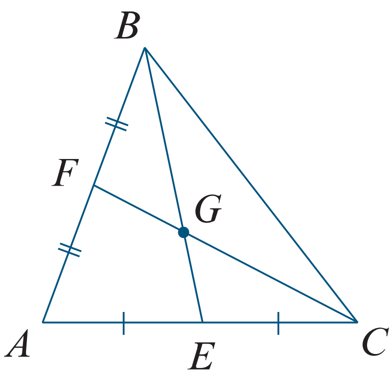
Question 14 (15 Marks)   Start a NEW Writing Booklet

(a) The curve with equation $y = x^5 - 3x^2 + x + 5$ is sketch below.
The curve passes through the points $A(-1, 0)$ and $B(1, 4)$.



Find the area of the shaded region bounded by the curve between A and B and the line segments AO and OB. **2**

(b) In ΔABC, E and F are the midpoints of AC and AB respectively.
BE and FC intersect at G.



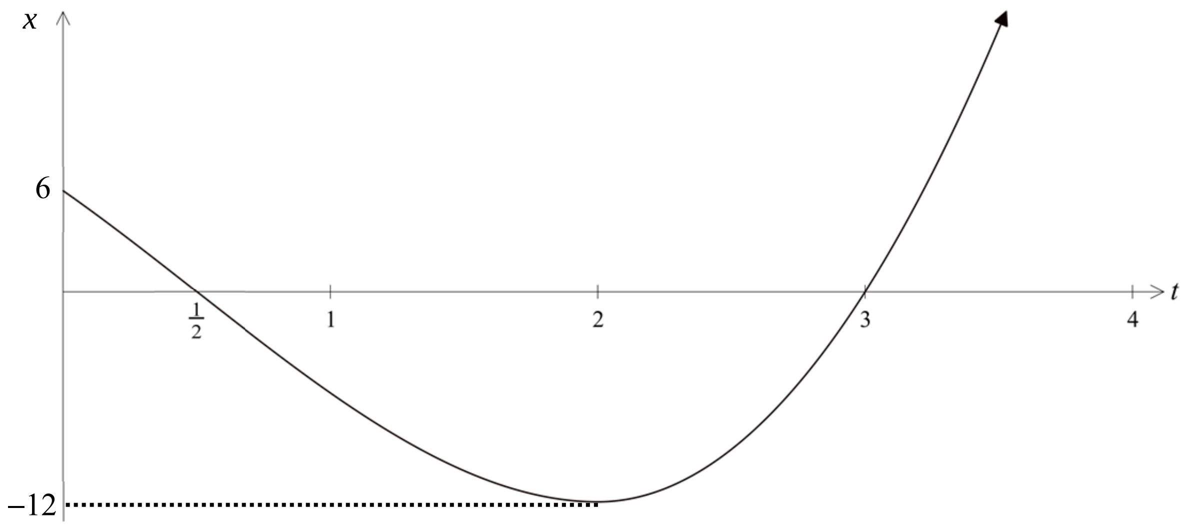
(i) State why $EF \parallel CB$. **1**

(ii) Prove that $\Delta BCG \parallel\!| \Delta EFG$. **2**

(iii) Hence show that $BG : GE = CG : GF = 2 : 1$. **2**

Question 14 continues on page 13

(c)



The displacement of a particle moving along a horizontal line is described by the diagram above.
The point $\left(\frac{1}{2}, 0\right)$ is the only point of inflexion and there is a turning point at $(2, -12)$.
The displacement x is in metres and the time t is in seconds.

(i) When is the particle stationary? **1**

(ii) What is the total distance travelled in the first 3 seconds? **1**

(iii) When is the acceleration of the particle positive? **1**

(d) Uhdam would like to save \$80 000 for a deposit on her first home.
She has decided to invest her net monthly salary of \$4500 at the beginning of each month.
She earns $4 \cdot 5$ % in interest per annum, compounded monthly.
Uhdam intends to withdraw \$*M* at the end of each month from her account for living expenses, immediately after the interest has been paid.

(i) Show that the amount of money at the end of the 2nd month following the second withdrawal of \$*M* is given by **2**

$\$4500\left(R^2 + R\right) - \$M\left(R+1\right)$, where $R = 1 + \frac{4 \cdot 5}{1200}$.

(ii) If Uhdam is to reach her goal in 6 years, show that **2**

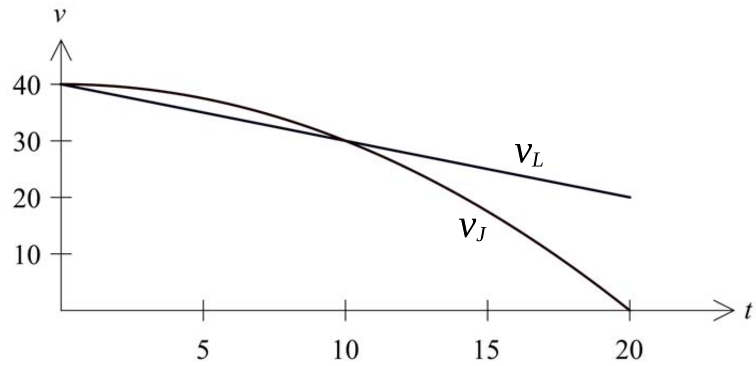
$$M = \frac{4500\left(R^{72} + R^{71} + \ldots + R\right) - 80\,000}{R^{71} + R^{70} + \ldots + R + 1}$$

(iii) Calculate the value of *M*. **1**
Leave your answer to the nearest integer.

End of Question 14

Question 15 (15 Marks) Start a NEW Writing Booklet

(a) James and Lauren are each speeding down a straight stretch of freeway and are side by side, when they spot a police car. They each brake.
Let t be measured in seconds from the time they spot the police car.
The velocity of Lauren's car during this braking phase is given by $v_L = 40 - t$ and the velocity of James's car during this phase is given by $v_J = 40 - \frac{1}{10}t^2$.



(i) When are the two cars level with one another during this braking phase? **2**

(ii) At what time is Lauren's car further ahead of James' car during this braking phase? **1**
Give reasons for your answer.

(b) A vessel initially contains 100 litres.
It is being emptied, and the rate of change of volume is given by $\frac{dV}{dt} = -\left(2 + \frac{20}{t+1}\right)$,
where V is the volume in litres after t minutes,

(i) What is the initial rate $\frac{dV}{dt}$? **1**

(ii) Find how many litres remain in the vessel after five minutes. **2**

(c) The roots of $x^2 - 2x - 5 = 0$ are α and β.

(i) Find the value of $\alpha^2 + \beta^2$ **2**

(ii) If $\alpha < \beta$, find the value of $\alpha - \beta$. **1**

Question 15 continues on page 15

Question 15 (continued)

(d) Atmospheric pressure is the pressure exerted by the air in the earth's atmosphere. It can be measured in kilopascals (kPa).

The average atmospheric pressure varies with altitude: the higher up one goes, the lower the pressure is.

Ellivlem in investigating the variation in pressure found the following:

Altitude (km)	0	1
Pressure (kPa)	$101 \cdot 3$	$89 \cdot 9$

Ellivlem suggests using the following function: $p = Ae^{-kh}$, where p is the pressure in kilopascals, and h is the altitude in kilometres.

(i) Show that $p = 101 \cdot 3e^{-0 \cdot 1194h}$. **3**

(ii) Use Ellivlem's function to estimate the atmospheric pressure at the top of Mount Everest (8848 metres). **1**

(iii) People sometimes experience a sensation in their ears when the pressure changes. **2**
This can happen when travelling in a fast lift in a tall building.
Experiments indicate that many people feel such a sensation if the pressure changes rapidly by 1 kilopascal or more.

Suppose that such a person steps into a lift that is close to sea level. Taking 3 m as a suitable approximation for the distance between two floors, estimate the number of floors that the person would need to travel in order to feel this sensation.

End of Question 15

Question 16 (15 Marks) Start a NEW Writing Booklet

(a) Consider the function $y = x^4 - 32x + 5$.

(i) Determine the nature of any stationary points. **2**

(ii) Find the coordinates of any points of inflexions. **2**

(b) Let $f(\theta) = \dfrac{2 - \cos\theta}{\sin\theta}, \quad 0 < \theta < \dfrac{\pi}{2}$.

(i) Show that $f'(\theta) = \dfrac{1 - 2\cos\theta}{\sin^2\theta}$. **2**

(ii) Show that the minimum value of $f(\theta)$ is $\sqrt{3}$. **3**

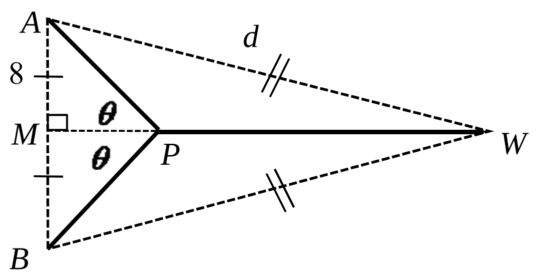
Question 16 continues on page 17

Question 16 (continued)

(c) The diagram below shows two towns A and B that are 16 km apart, and each at a distance of d km from a water well at W.

Let M be the midpoint of AB, P be a point on the line segment MW, and $\theta = \angle APM = \angle BPM$.

The two towns are to be supplied with water from W, via three straight water pipes: PW, PA and PB as shown below.



(i) Show that the total length of the water pipe L is given by **3**

$$L = 8f(\theta) + \sqrt{d^2 - 64}$$

where $f(\theta)$ is given in part (b) above.

NB For this to occur $\frac{8}{d} \le \sin\theta \le 1$. (Do NOT prove this)

(ii) Find the minimum value of L if $d = 20$. **1**

(iii) If $d = 9$, show that the minimum value of L cannot be found by using the same methods as used in part (ii). **2**
Explain briefly how to find the minimum value of L in this case.

End of paper

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STANDARD INTEGRALS

$$\int x^n\,dx = \frac{1}{n+1}x^{n+1}, \quad n \neq -1; \quad x \neq 0, \text{ if } n < 0$$

$$\int \frac{1}{x}\,dx = \ln x, \quad x > 0$$

$$\int e^{ax}\,dx = \frac{1}{a}e^{ax}, \quad a \neq 0$$

$$\int \cos ax\,dx = \frac{1}{a}\sin ax, \quad a \neq 0$$

$$\int \sin ax\,dx = -\frac{1}{a}\cos ax, \quad a \neq 0$$

$$\int \sec^2 ax\,dx = \frac{1}{a}\tan ax, \quad a \neq 0$$

$$\int \sec ax \tan ax\,dx = \frac{1}{a}\sec ax, \quad a \neq 0$$

$$\int \frac{1}{a^2+x^2}\,dx = \frac{1}{a}\tan^{-1}\frac{x}{a}, \quad a \neq 0$$

$$\int \frac{1}{\sqrt{a^2-x^2}}\,dx = \sin^{-1}\frac{x}{a}, \quad a > 0, \quad -a < x < a$$

$$\int \frac{1}{\sqrt{x^2-a^2}}\,dx = \ln\left(x+\sqrt{x^2-a^2}\right), \quad x > a > 0$$

$$\int \frac{1}{\sqrt{x^2+a^2}}\,dx = \ln\left(x+\sqrt{x^2+a^2}\right)$$

NOTE : $\ln x = \log_e x, \quad x > 0$

NORTH SYDNEY GIRLS HIGH SCHOOL

2013

TRIAL HSC EXAMINATION

Mathematics

Sample Solutions

Section I

1. (A) (B) (C) ●

2. (A) ● (C) (D)

3. ● (B) (C) (D)

4. ● (B) (C) (D)

5. (A) (B) ● (D)

6. (A) (B) (C) ●

7. (A) (B) ● (D)

8. (A) (B) ● (D)

9. ● (B) (C) (D)

10. (A) (B) (C) ●

Section I Worked Solutions

1 Which of the following represents $\log_e\left(\frac{10^3}{e^{10}-10}\right)$, evaluated to four significant figures?

(A) -1.3427
(B) -1.343
(C) -3.0918
(D) -3.092

2 What is the equation of the directrix of the parabola $x^2 = -8y$?

(A) $x = 2$
(B) $y = 2$
(C) $x = 8$
(D) $y = -8$

3 What is $8^3 \times 6^{\frac{1}{2}} \div 32^{\frac{3}{2}}$ in simplest form?

$$8^3 \times 6^{\frac{1}{2}} \div 32^{\frac{3}{2}} = \frac{\left(2^3\right)^3 \times \left(2\times 3\right)^{\frac{1}{2}}}{\left(2^5\right)^{\frac{3}{2}}} = 2^{9+\frac{1}{2}-7\frac{1}{2}} \times 3^{\frac{1}{2}} = 2^2 \times 3^{\frac{1}{2}}$$

(A) $4\sqrt{3}$
(B) $2\sqrt{3}$
(C) $3\sqrt{2}$
(D) $4\sqrt{2}$

4 What is the sum of the first 12 terms of the following arithmetic series?

$$-20-13-6+1+\ldots$$

$$a=-20,\ d=7,\ S_{12}=\frac{12}{2}\left[2\times(-20)+(12-1)\times 7\right]=$$

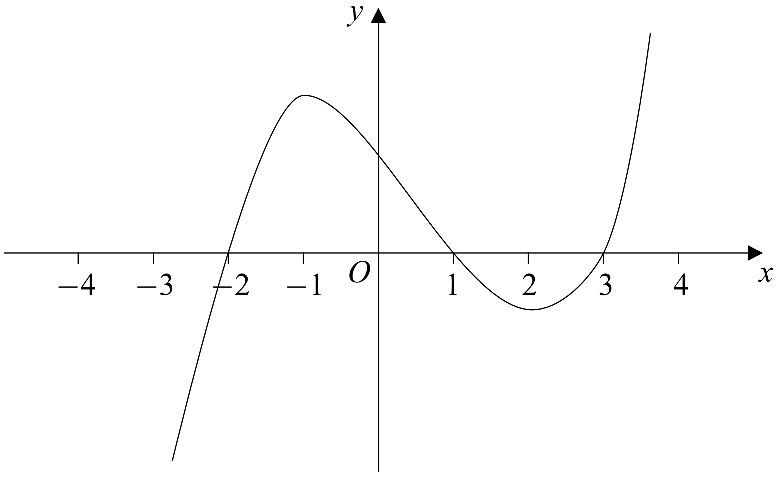
(A) 222
(B) -702
(C) 264
(D) -744

5 The quadratic equation $5x^2 - 7x + 1 = 0$ has roots α and β.
What is the value of $\frac{3}{\alpha}+\frac{3}{\beta}$?

$$\alpha+\beta=\frac{7}{5},\ \alpha\beta=\frac{1}{5},\ \frac{3}{\alpha}+\frac{3}{\beta}=\frac{3(\alpha+\beta)}{\alpha\beta}=\frac{3\times\frac{7}{5}}{\frac{1}{5}}=21$$

(A) -21
(B) 7
(C) 21
(D) -7

6 The graph of $y = f(x)$ is drawn below. It has a maximum turning point at (–1, 10).



What are the coordinates of the maximum point of the curve $y = \frac{1}{2} f(x+1)$?

The transformed graph has been shifted to the left by 1 unit and the y-values halved.

$\therefore (-1, 10) \to \left(-1-1, \frac{1}{2} \times 10\right) = (-2, 5)$

(A) $(0, 5)$

(B) $\left(-\frac{1}{2}, 5\right)$

(C) (–1, 5)

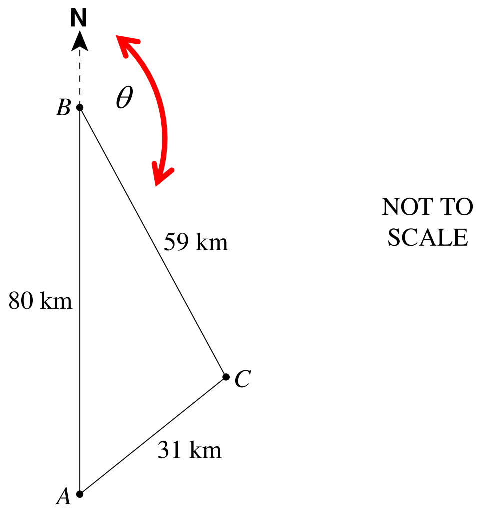
(D) (–2, 5)

7 In the diagram below, Town *B* is 80 km due north of town *A* and 59 km from Town *C*. Town *A* is 31 km from Town *C*.

$\cos \angle ABC = \dfrac{80^2 + 59^2 - 31^2}{2 \times 80 \times 59} = \dfrac{223}{236}$

$\angle ABC \doteqdot 19°$

$\therefore \theta \doteqdot 161$



What is the bearing of Town *C* from Town *B*?

(A) 019° (B) 122° (C) 161° (D) 341°

8 If $y = 3\cos^4 x$, what is $\dfrac{dy}{dx}$?

$y = 3\cos^4 x = 3(\cos x)^4$

$\therefore \dfrac{dy}{dx} = 12(\cos x)^3 \times (-\sin x)$

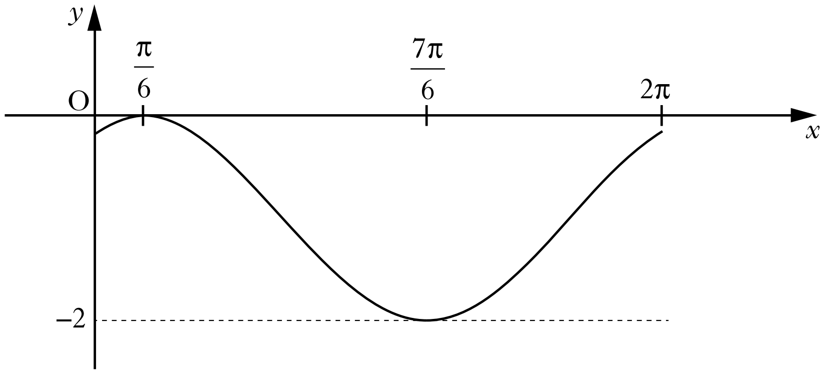
(A) $12\cos^3 x \sin x$

(B) $12\cos^3 x$

(C) $-12\cos^3 x \sin x$

(D) $-12\sin^3 x$

9 What is the equation of this curve?



The graph is $y = \cos x$ shifted to the right by $\dfrac{\pi}{6}$ units and shifted down 1 unit.

(A) $y = \cos\left(x - \dfrac{\pi}{6}\right) - 1$

(B) $y = \cos\left(x - \dfrac{\pi}{6}\right) + 1$

(C) $y = \cos\left(x + \dfrac{\pi}{6}\right) - 1$

(D) $y = \cos\left(x + \dfrac{\pi}{6}\right) + 1$

10 In the diagram below, $AB \parallel CD$, EF bisects $\angle BEG$ and GF bisects $\angle EGD$. What is the size of $\angle EFG$?

Let $\angle BEF = \alpha$ and $\angle FGD = \beta$.

$\therefore \angle FEG = \alpha$ $\quad$ (EF bisects $\angle BEG$)

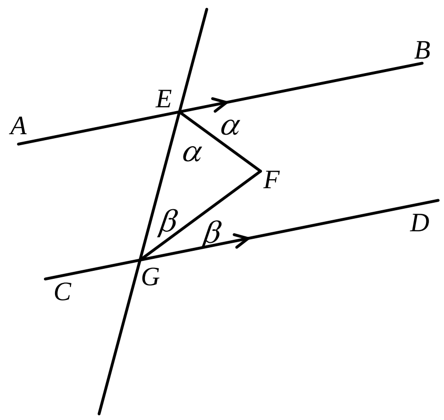
Similarly, $\angle FGE = \beta$

$2\alpha + 2\beta = 180°$ $\quad$ (cointerior angles, $AB \parallel CD$)

$\therefore \alpha + \beta = 90°$

$\angle EFG + \alpha + \beta = 180°$ $\quad$ (angle sum ΔEFG)

$\therefore \angle EFG = 90°$



(A) 30° (B) 45° (C) 60° (D) 90°

Section II

Question 11

(a) $$\frac{2}{2-\sqrt{3}} = \frac{2}{2-\sqrt{3}} \times \frac{2+\sqrt{3}}{2+\sqrt{3}}$$

$$= \frac{2\left(2+\sqrt{3}\right)}{4-3}$$ **1**

$$= 4+2\sqrt{3}$$

(b) $x-2=\sqrt{3x-2}$

$\therefore \left(x-2\right)^2 = 3x-2$

$\therefore x^2-4x+4=3x-2$ **3**

$\therefore x^2-7x+6=0$

$\therefore \left(x-6\right)\left(x-1\right)=0$

$\therefore x=1, 6$

With $x-2=\sqrt{3x-2}$, the RHS ≥ 0 for all $x \geq \frac{2}{3}$.
NB $x = 1$ is an invalid solution as LHS < 0 on substitution.
$\therefore x=6$ only.

(c) (i) $T_1 = a = 256, T_4 = ar^3 = 2048$ **1**

$$\frac{T_4}{T_1} = \frac{ar^3}{a} = \frac{2048}{256}$$

$\therefore r^3 = 8$

$\therefore r = 2$

(ii) $S_n = 261\,888$ **2**

$$S_n = \frac{a\left(r^n-1\right)}{r-1}$$

$$\therefore \frac{256\left(2^n-1\right)}{2-1} = 261\,888$$

$$\therefore 2^n - 1 = \frac{261\,888}{256}$$

$$\therefore 2^n = \frac{261\,888}{256} + 1 = 1024 = 2^{10}$$

$$\therefore n = 10 \qquad \left(\text{or } \frac{\ln 1024}{\ln 2} = 10\right)$$

Question 11 continued

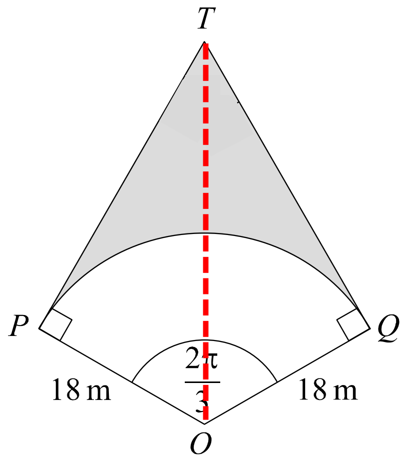
(d) (i) $$\frac{dy}{dx} = 10\left(4x^2+3x+2\right)^9 \times (8x+3)$$ **2**

$$= 10(8x+3)\left(4x^2+3x+2\right)^9$$

(ii) $$\frac{dy}{dx} = x^2 \times \sec^2 x + 2x \times \tan x$$ **2**

$$= x^2 \sec^2 x + 2x \tan x$$

(e)



(i) $$\text{Area} = \frac{1}{2} \times 18^2 \times \frac{2\pi}{3}$$ **1**

$$= 108\pi \text{ m}^2$$

(ii) Join OT **1**

$$\therefore \angle TOP = \frac{\pi}{3}.$$

$$\tan \angle TOP = \frac{PT}{OP}$$

$$\therefore \tan \frac{\pi}{3} = \frac{PT}{18}$$

$$\therefore PT = 18 \tan \frac{\pi}{3} = 18\sqrt{3}$$

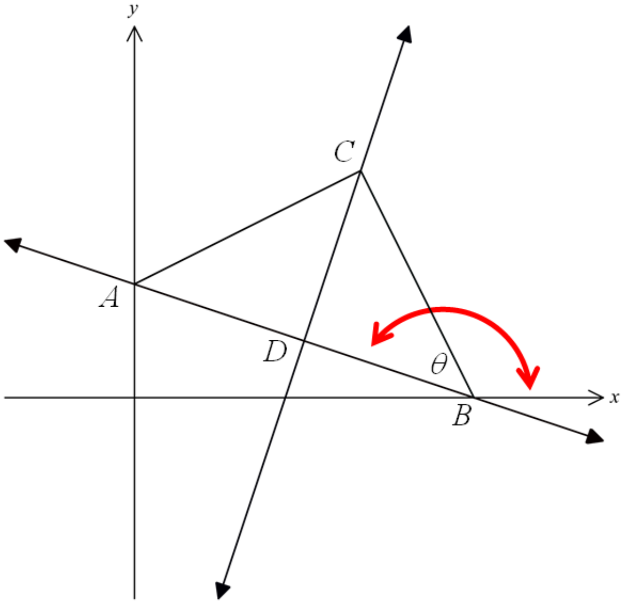
(iii) Area $OPTQ$ = 2 × area Δ TOP = $2 \times \left(\frac{1}{2} \times 18 \times 18\sqrt{3} \right) = 324\sqrt{3} \text{ m}^2$ **2**

Shaded area = Area $OPTQ$ – sector OPQ

$= 324\sqrt{3} - 108\pi \doteqdot 222 \text{ m}^2$ (3 sig fig)

Question 12

(a)



(i) Let θ be the angle that *AB* makes with the positive direction of the *x*-axis **2**

$m_{AB} = -\dfrac{1}{3}$

$\theta = 180° - \tan^{-1}\dfrac{1}{3} \doteqdot 162° \qquad (161°34')$

(ii) *D* (3, 1)

As $CD \perp AB$ then $m_{CD} = 3$ **2**

$\therefore y - 1 = 3(x - 3)$

$\therefore y - 1 = 3x - 9$

$\therefore 3x - y - 8 = 0$

(iii) If $C(4, k)$, show that $k = 4$. **1**

Substitute $(4, k)$ into the equation of *CD*.

$\therefore 3 \times 4 - k - 8 = 0$

$\therefore k = 4$

(iv) $AB = \sqrt{2^2 + 6^2} = \sqrt{40}$ **2**

Using $d = \dfrac{|\ Ax_1 + By_1 + C\ |}{\sqrt{A^2 + B^2}}$ to get *CD*.

$CD = \dfrac{|4 + 3 \times 4 - 6|}{\sqrt{1^2 + 3^2}}$

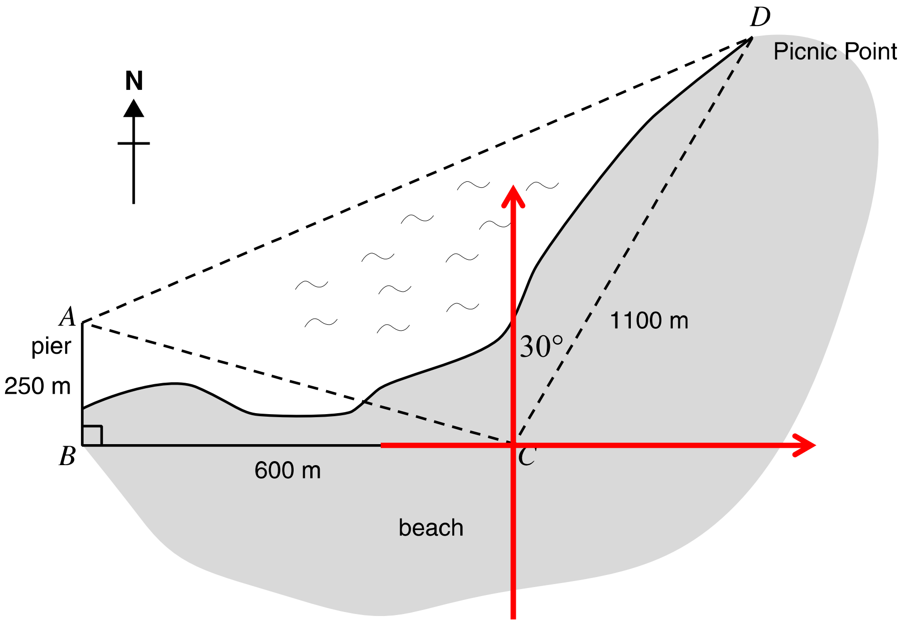
$= \dfrac{10}{\sqrt{10}}$

[Alternatively: using the distance formula $CD = \sqrt{(4-3)^2 + (4-1)^2} = \sqrt{10}$]

Area $\Delta ABC = \dfrac{1}{2} \times \sqrt{40} \times \dfrac{10}{\sqrt{10}} = 10 \text{ u}^2$

Question 12 continued

(b)



(i) By Pythagoras' Theorem: $250^2 + 600^2 = 650^2$.
So $AC = 650$

(ii) $\tan \angle BCA = \dfrac{250}{600}$

$\therefore \angle BCA \doteqdot 23°$

(iii) $\angle ACD = 90° - \angle BCA + 30° \doteqdot 97°$

Using the cosine rule in $\Delta\, ACD$

$$AD^2 = 650^2 + 1100^2 - 2 \times 650 \times 1100 \times \cos 97°$$
$$= 1806773 \cdot 161...$$

$\therefore AD \doteqdot 1344$ m

(c) (i) $\dfrac{d}{dx}\left(4x^3 - 6x + 1\right) = 12x^2 - 6$ **1**

$$= 6\left(2x^2 - 1\right)$$

(ii) $$\int_2^3 \frac{2x^2 - 1}{4x^3 - 6x + 1}\,dx = \frac{1}{6}\int_2^3 \frac{6\left(2x^2 - 1\right)}{4x^3 - 6x + 1}\,dx$$

$$= \left[\frac{1}{6}\ln\left(4x^3 - 6x + 1\right)\right]_2^3$$

$$= \frac{1}{6}\left(\ln 91 - \ln 21\right)$$ **2**

$$= \frac{1}{6}\ln\left(\frac{91}{21}\right) \quad \left[= \frac{1}{6}\ln\left(\frac{13}{3}\right)\right]$$

Question 13

(a) $\text{LHS} = \sec^2\theta + \text{cosec}^2\theta$

$$= \frac{1}{\cos^2\theta} + \frac{1}{\sin^2\theta}$$

$$= \frac{\sin^2\theta + \cos^2\theta}{\cos^2\theta\sin^2\theta}$$ **2**

$$= \frac{1}{\cos^2\theta\sin^2\theta}$$

$$= \sec^2\theta\,\text{cosec}^2\theta = \text{RHS}$$

(b) (i) $\left(\sqrt{3}u - 1\right)\left(u - \sqrt{3}\right) = \sqrt{3}u^2 - 4u + \sqrt{3}$ **1**

(ii) From (i) $\sqrt{3}\tan^2\theta - 4\tan\theta + \sqrt{3} = \left(\sqrt{3}\tan\theta - 1\right)\left(\tan\theta - \sqrt{3}\right)$ **2**

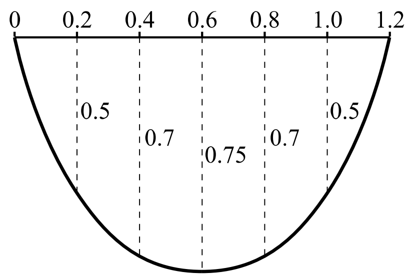
$\sqrt{3}\tan^2\theta - 4\tan\theta + \sqrt{3} = 0 \Rightarrow \left(\sqrt{3}\tan\theta - 1\right)\left(\tan\theta - \sqrt{3}\right) = 0$

$\therefore \tan\theta = \frac{1}{\sqrt{3}}, \sqrt{3}$

$\therefore \theta = \frac{\pi}{6}, \frac{7\pi}{6}, \frac{\pi}{3}, \frac{4\pi}{6}$

(c) $h = 0.2$ **3**

x	y	w (weight)	yw
0	0	1	0
0.2	0.5	2	1.0
0.4	0.7	2	1.4
0.6	0.75	2	1.5
0.8	0.7	2	1.4
1.0	0.5	2	1.0
1.2	0	1	0
		$\Sigma\, yw$	6.3



Cross sectional area $\doteqdot \frac{h}{2} \times 6.3 = \frac{0.2}{2} \times 6.3 = 0.63\,\text{m}^2$

$H = 50$

$V = AH \doteqdot 0.63 \times 50 = 31.5\,\text{m}^3$

Question 13 (continued)

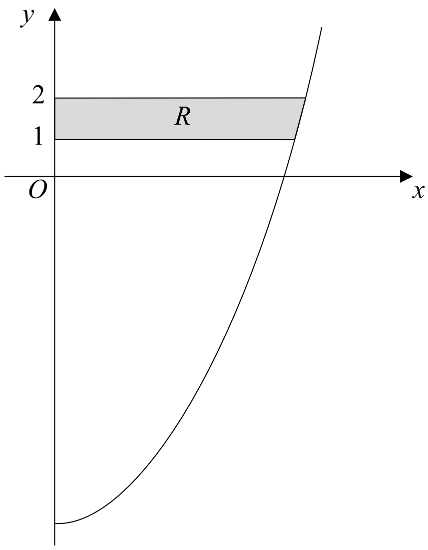
(d) $V = \pi \int_1^2 x^2 dy$

$= \pi \int_1^2 (y+9)\, dy$

$= \pi \left[9y + \frac{1}{2} y^2 \right]_1^2$ **2**

$= \pi \left[(18+2) - \left(9 + \frac{1}{2} \right) \right]$

$= 10.5\pi$ cu



(e) (i) $a = \frac{dv}{dt}$ **1**

$= 12t + 32e^{-4t}$

$t = 0$, $a = 32$

(ii) $t = 0$, $v = -8 + 9 = 1$

So it is initially moving to the right

(ii) $t = 0$, $x = 0$. **2**

$x = \int v \ dt$

$= 2t^3 + 2e^{-4t} + 9t + c$

Substitute $t = 0$, $x = 0$.

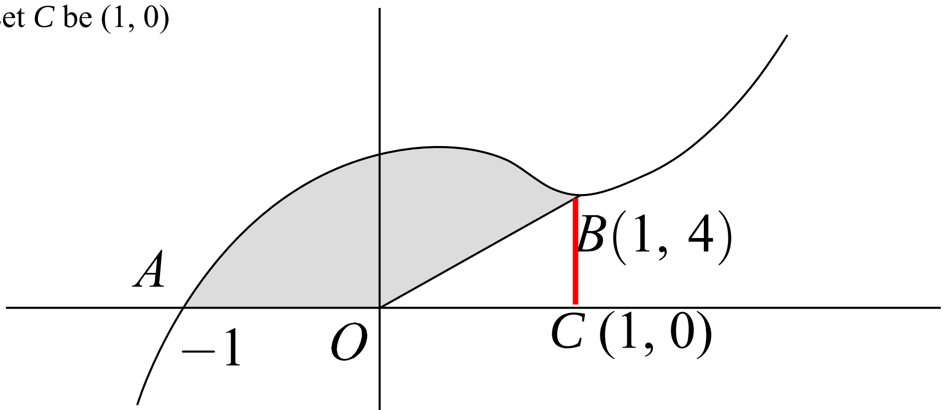
$0 = 0 + 2 + 0 + c$

$\therefore c = -2$

$\therefore x = 2t^3 + 2e^{-4t} + 9t - 2$

Question 14

(a) Let C be $(1, 0)$ **2**



Shaded area $= \int_{-1}^{1} \left(x^5 - 3x^2 + x + 5\right) dx -$ area $\Delta\, BOC$

$$\int_{-1}^{1} \left(x^5 - 3x^2 + x + 5\right) dx = \left[\frac{1}{6}x^6 - x^3 + \frac{1}{2}x^2 + 5x\right]_{-1}^{1}$$

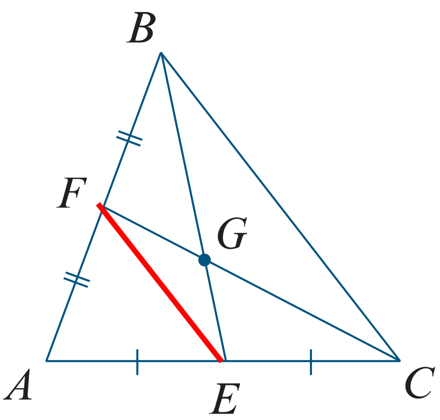
$$= \left(\frac{1}{6} - 1 + \frac{1}{2} + 5\right) - \left(\frac{1}{6} + 1 + \frac{1}{2} - 5\right)$$

$$= 8$$

Area $\Delta\, BOC = \frac{1}{2} \times 1 \times 4 = 2$

$\therefore$ Shaded area $= 6\ \text{u}^2$

(b)



(i) $EF \parallel CB$ (join of midpoints) **1**

NB $EF = \frac{1}{2}CB$ as well i.e. $CB : EF = 2 : 1$

(ii) In $\Delta\, BCG$ and $\Delta\, EFG$ **2**

$\angle FGE = \angle CGB$ (vertically opposite)

$\angle EFG = \angle GCB$ (alternate angles, $EF \parallel CB$)

$\therefore \Delta BCG \,|||\, \Delta EFG$ (equiangular)

Question 14 (continued)

(b) (iii) $BG:GE = CG:GF = CB:EF$ (matching sides of similar triangles) **2**

From (i), $CB:EF = 2:1$

$\therefore BG:GE = CG:GF = 2:1$

(c) (i) The particle is stationary when $v = \frac{dx}{dt} = 0$. **1**

$\therefore t = 2$

(ii) Distance = 6 + 12 + 12 = 30 m **1**

(iii) $a = \frac{d^2x}{dt^2}$ **1**

As $a = 0$ when $t = \frac{1}{2}$, then $a > 0$ when the graph is concave up i.e. $t > \frac{1}{2}$

(d) (i) Let A_n be the amount of money left in her account after n months. **2**

Let $R = 1 + \frac{4 \cdot 5}{1200}$

$$A_1 = 4500R - M$$

$$\begin{aligned} A_2 &= (4500 + A_1)R - M \\ &= (4500 + 4500R - M)R - M \\ &= 4500(R + R^2) - MR - M \\ &= 4500(R + R^2) - M(1 + R) \end{aligned}$$

(ii) 6 years = 72 months. **2**

Following the pattern in (i): $A_{72} = 4500(R + R^2 + ... + R^{72}) - M(1 + R + ... + R^{71})$

The goal is $A_{72} = 80\,000$.

$$\therefore 4500(R + R^2 + ... + R^{72}) - M(1 + R + ... + R^{71}) = 80\,000$$

$$\therefore M = \frac{4500(R^{72} + R^{71} + ... + R) - 80\,000}{R^{71} + R^{70} + ... + R + 1}$$

(d) (iii) From (ii), $M = \frac{4500(R^{72} + R^{71} + ... + R) - 80\,000}{R^{71} + R^{70} + ... + R + 1}$ **1**

$$\therefore M = \frac{4500\left[\frac{R(R^{72} - 1)}{R - 1}\right] - 80\,000}{\frac{(R^{72} - 1)}{R - 1}}$$

$$= 4500R - 80\,000\left(\frac{R - 1}{R^{72} - 1}\right)$$

$$= 3547$$

So Uhdam will take out \$3547.

Question 15

(a) (i) Let T be the first time, after the start, when the two cars are level. **2**

$$\therefore \int_0^T v_J \, dt = \int_0^T v_L \, dt$$

$$\therefore \left[40t - \frac{1}{30}t^3\right]_0^T = \left[40t - \frac{1}{2}t^2\right]_0^T$$

$$\therefore 40T - \frac{1}{30}T^3 = 40T - \frac{1}{2}T^2$$

$$\therefore \frac{1}{30}T^3 - \frac{1}{2}T^2 = 0$$

$$\therefore \frac{1}{30}T^2\left(T - 15\right) = 0$$

$$\therefore T = 0, 15$$

$$\therefore T = 15, T > 0$$

(ii) $t > 15$ **1**

In fact for $15 < t \le 40$, Lauren has a higher velocity than James.
Since at t = 15 they are level, then after that Lauren will be ahead.

(b) (i) $t = 0,\ \dfrac{dV}{dt} = -\left(2 + 20\right) = -22$ L/min **1**

i.e. it is emptying at 22 L/min.

(ii) $t = 5, V = ?$ **2**

$$\frac{dV}{dt} = -\left(2 + \frac{20}{t+1}\right)$$

$$\therefore V = -2t - 20\ln\left(t+1\right) + C$$

Substitute $t = 0$, $V = 100$

$$\therefore 100 = 0 - 20\ln\left(1\right) + C$$

$$\therefore C = 100$$

$$\therefore V = -2t - 20\ln\left(t+1\right) + 100$$

$$\therefore t = 5, V = -2\times 5 - 20\ln\left(5+1\right) + 100 = 90 - 20\ln 6 \doteqdot 54\cdot 2 \text{ L}$$

(c) (i) $\alpha + \beta = 2, \alpha\beta = -5$ **2**

$$\alpha^2 + \beta^2 = \left(\alpha + \beta\right)^2 - 2\alpha\beta$$

$$= 2^2 - 2\times\left(-5\right)$$

$$= 14$$

Question 15 (continued)

(c) (ii) $(\alpha-\beta)^2 = \alpha^2+\beta^2-2\alpha\beta$ **1**

$= (\alpha+\beta)^2 - 4\alpha\beta$

$= 2^2 + 4\times 5$

$= 24$

As $\alpha-\beta<0$, then $\alpha-\beta = -\sqrt{24} = -2\sqrt{6}$

(d) (i) $h=0,\ p=101\cdot 3 = Ae^0$ **3**

$\therefore A = 101\cdot 3$

$\therefore p = 101\cdot 3e^{-kh}$

Substitute $h = 1,\ p = 89\cdot 9$

$\therefore 89\cdot 9 = 101\cdot 3e^{-k}$

$\therefore e^{-k} = \dfrac{89\cdot 9}{101\cdot 3}$

$\therefore -k = \ln\left(\dfrac{89\cdot 9}{101\cdot 3}\right)$

$\therefore k = -\ln\left(\dfrac{89\cdot 9}{101\cdot 3}\right) = \ln\left(\dfrac{101\cdot 3}{89\cdot 9}\right) \doteqdot 0\cdot 1194$

$\therefore p = 101\cdot 3e^{-0\cdot 1194h}$.

(ii) $h = 8\cdot 848,\ p = 101\cdot 3e^{-0\cdot 1194\times 8\cdot 848} \doteqdot 35\cdot 2$ **1**

At the top of Everest, the pressure is $35\cdot 2$ kPa.

(iii) To get a difference of 1 kPa going up in an elevator means solving **2**

$p = 100\cdot 3 = 101\cdot 3e^{-0\cdot 1194h}$.

$\therefore e^{-0\cdot 1194h} = \dfrac{100\cdot 3}{101\cdot 3}$

$\therefore -0\cdot 1194h = \ln\left(\dfrac{100\cdot 3}{101\cdot 3}\right)$

$\therefore h = \dfrac{\ln\left(\frac{100\cdot 3}{101\cdot 3}\right)}{-0\cdot 1194} \doteqdot 0\cdot 0830880761\text{ km}$

$\therefore h \doteqdot 83\cdot 0880761\text{ m}$

As the height of each floor is 3 m, then $\dfrac{h}{3} \doteqdot \dfrac{83\cdot 0880761}{3}\text{ m} \doteqdot 27\cdot 7\text{ m}$

So 28 floors will be needed to get the 1 kPa change.

Question 16

(a) (i) Stationary points occur when $\frac{dy}{dx} = 0$ **2**

$$\frac{dy}{dx} = 4x^3 - 32$$
$$\therefore 4x^3 - 32 = 0$$
$$\therefore 4\left(x^3 - 8\right) = 0$$
$$\therefore x = 2$$
$$\frac{d^2y}{dx^2} = 12x^2$$

Substitute $x = 2$ into $\frac{d^2y}{dx^2}$.

$$\frac{d^2y}{dx^2} = 12 \times 2^2 = 48 > 0$$

$$x = 2,\ y = 2^4 - 32 \times 2 + 5 = -43$$

So at (2, –43) there is a minimum turning point.
This is a global minimum as there are no other stationary points.

(ii) Points of inflexion occur at a change in concavity **2**

$\frac{d^2y}{dx^2} = 12x^2$ is always positive except at $x = 0$, so there are no points of inflexion.

Question 16 (continued)

(b) (i) $$f'(\theta) = \frac{\sin\theta \times (\sin\theta) - (2-\cos\theta)\cos\theta}{\sin^2\theta}$$ **2**

$$= \frac{\sin^2\theta + \cos^2\theta - 2\cos\theta}{\sin^2\theta}$$

$$= \frac{1-2\cos\theta}{\sin^2\theta}$$

(ii) The minimum value of $f(\theta)$ occurs when $f'(\theta) = 0$ **3**

$$\therefore \frac{1-2\cos\theta}{\sin^2\theta} = 0$$

$$\therefore 1-2\cos\theta = 0$$

$$\therefore \cos\theta = \frac{1}{2}$$

$$\therefore \theta = \frac{\pi}{3} \qquad \left(0 < \theta \leq \frac{\pi}{2}\right)$$

Only need to check the numerator as the denominator is always positive.

θ	1	$\frac{\pi}{3}\ (\doteqdot 1\cdot 05)$	$1\cdot 1$
$f'(\theta)$	$-0\cdot 1$	0	$0\cdot 1$

So there is a minimum at $\theta = \frac{\pi}{3}$.

$$f\left(\frac{\pi}{3}\right) = \frac{2-\cos\frac{\pi}{3}}{\sin\frac{\pi}{3}}$$

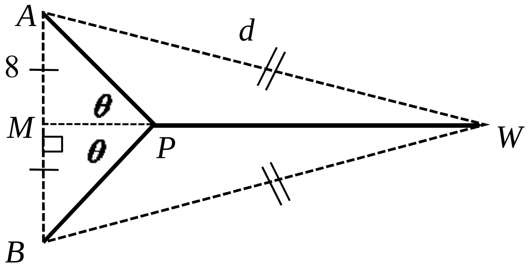
$$= \frac{2-\frac{1}{2}}{\frac{\sqrt{3}}{2}}$$

$$= \frac{\frac{3}{2}}{\frac{\sqrt{3}}{2}}$$

$$= \frac{3}{\sqrt{3}}$$

$$= \sqrt{3}$$

(c) (i) $L = AP + BP + PW$ **3**



$\Delta\, APB$ is isosceles (SAS)

$\therefore\, AP = BP$ and $L = 2AP + PW$.

$PW = MW - MP$

$$= \sqrt{d^2 - 64} - \frac{8}{\tan\theta}$$

$$AP = \frac{8}{\sin\theta}$$

$$L = 2\times\frac{8}{\sin\theta} + \sqrt{d^2-64} - \frac{8}{\tan\theta}$$

$$= 2\times\frac{8}{\sin\theta} + \sqrt{d^2-64} - \frac{8\cos\theta}{\sin\theta}$$

$$= 8\times\frac{2-\cos\theta}{\sin\theta} + \sqrt{d^2-64}$$

$$= 8f(\theta) + \sqrt{d^2-64}$$

NB Why $\frac{8}{d} \le \sin\theta \le 1$? This is to ensure that in the diagram $\angle APM > \angle AWP$ and so that P is "inside" $\Delta\, ABM$.

(ii) $L = 8f(\theta) + \sqrt{d^2-64}$. **1**

$$\therefore L_{\min} = 8\times\sqrt{3} + \sqrt{20^2-64} = 8\sqrt{3} + \sqrt{336}$$

$$= 8\sqrt{3} + 4\sqrt{21}$$

[Why? $f\left(\frac{\pi}{3}\right) = \sqrt{3}$ is the minimum of $f(\theta)$, and $\frac{8}{20} \le \sin\frac{\pi}{3} = \frac{\sqrt{3}}{2} \le 1$.]

(iii) $d = 9$ does not satisfy $\frac{8}{d} \le \sin\theta \le 1$ i.e. $\frac{8}{9} \not\le \sin\frac{\pi}{3} = \frac{\sqrt{3}}{2} \le 1$. **2**

So P is "outside" $\Delta\, ABM$ as $\angle APM < \angle AWP$.

To find $L_{\min}$ test the boundaries: i.e. $\sin\theta = \frac{8}{9}$ and $\sin\theta = 1$

i.e. $\theta = \sin^{-1}\left(\frac{8}{9}\right) \doteqdot 1\cdot095$ and $\theta = \frac{\pi}{2}$.

$$\theta = \sin^{-1}\left(\tfrac{8}{9}\right): \quad L = 8\times\frac{2-\cos(1\cdot095)}{\sin^2(1\cdot095)} + \sqrt{9^2-64} \doteqdot 15\cdot6 + \sqrt{65}$$

$$\theta = \frac{\pi}{2}: \quad L = 8\times\frac{2-\cos\frac{\pi}{2}}{\sin^2\frac{\pi}{2}} + \sqrt{9^2-64} = 16 + \sqrt{65}$$

$$\therefore L_{\min} \doteqdot 15\cdot6 + \sqrt{65}$$

End of Solutions