



Oakhill

COLLEGE

Mathematics Extension 2

2009

Year 12 Trial Higher School Certificate Examination

Assessment Value: 40 %

General Instructions

- Reading time – 5 minutes
- Writing time – 3 hours
- Write using black or blue pen
- Board approved calculators may be used
- A table of standard integrals is provided at the back of this paper
- All necessary working should be shown in every question

Total Marks – 120

- All questions should be attempted
- All questions are of equal value

Examiner: Mrs P. O'Reilly
Assessor: Mr D. Findlay
Reviewer: Mr T. Attard
Marker: Mrs E. Batfay

QUESTION 1 (15 MARKS) Start question in new booklet

- | | | Marks |
|---|---|-------|
| (a) | (i) Find real numbers a and b such that: | 2 |
| $\frac{5x^2 - 3x + 1}{(x^2 + 1)(x - 2)} = \frac{ax + 1}{x^2 + 1} + \frac{b}{x - 2}$ | | |
| | (ii) Hence, find $\int \frac{5x^2 - 3x + 1}{(x^2 + 1)(x - 2)} dx$. | 2 |
| (b) | (i) Show that $\tan^3 \theta = \sec^2 \theta \tan \theta - \tan \theta$. | 1 |
| | (ii) Hence, find $\int \tan^3 \theta d\theta$. | 2 |
| (c) | Find $\int \frac{dx}{\sqrt{x^2 - 4x + 2}}$. | 3 |
| (d) | (i) Let $I_n = \int_1^e x(\ln x)^n dx$, $n = 0, 1, 2, 3, \dots$
Using integration by parts, show that $I_n = \frac{e^2}{2} - \frac{n}{2} I_{n-1}$ $n = 1, 2, 3, \dots$ | 2 |
| | (ii) The area bounded by the curve $y = \sqrt{x}(\ln x)^2$, $x \geq 1$, the x -axis and the line $x=e$ is rotated about the x -axis through 2π radians.

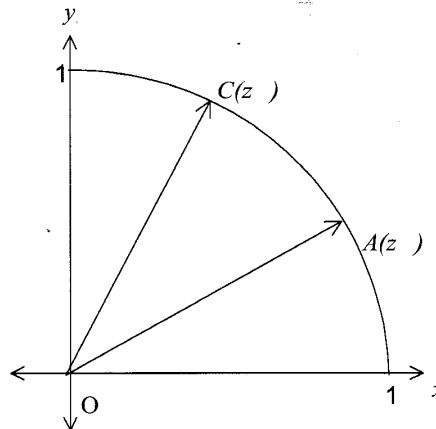
Find the exact volume of the solid of revolution formed. | 3 |

QUESTION 2 (15 MARKS) Start question in new booklet

(a) Express $z = \frac{7+4i}{3-2i}$ in the form $a+ib$, where a and b are real. Marks
2

(b) On an Argand diagram sketch the locus of the points representing the complex number z where $|z-3-i| = \sqrt{10}$. 3
Hence, find the greatest value of $|z|$ subject to this condition.

(c)



In the Argand diagram above, the two points A and C lie on the circumference of the circle with centre the origin of radius 1. They represent the complex numbers z_1 and z_2 respectively.

- (i) Copy the diagram into your answer sheet. Mark on your diagram the position of the point B that represents the complex number $z_1 + z_2$. 1
- (ii) Explain why AC is perpendicular to OB . 1

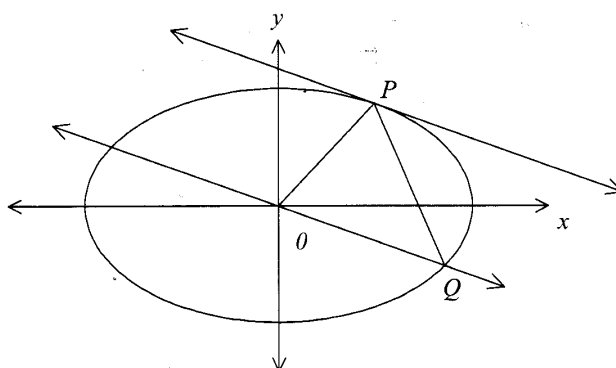
Question 2 continues over the page

Question 2 Continued

Marks

- (d) In the diagram below, $P(a \cos \theta, b \sin \theta)$ is a point on the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, where P lies in the first quadrant.

A straight line through the origin parallel to the tangent at P meets the ellipse at the point Q , where P and Q both lie on the same side of the y -axis.



- (i) Prove that the equation of the line OQ is $x b \cos \theta + y a \sin \theta = 0$. **2**
- (ii) Find the coordinates of the point Q given that Q lies in the fourth quadrant. **3**
- (iii) Prove that the area of $\triangle OPQ$ is independent of the position of P . **3**

QUESTION 3 (15 MARKS) Start question in new booklet

- (a) (i) Show that the area of a regular hexagon of sides s units is given by Marks
1

$$A = \frac{3\sqrt{3}s^2}{2} \text{ units}^2.$$

- (ii) The diagram below illustrates a dome tent. When erected, the base is a regular hexagon which measures 2 m from one corner to the adjacent corner (internal measurements). 2
Flexible exterior poles extend between opposite corners in semi-circular arcs to support the tent.

By taking slices parallel to the base, find the volume enclosed by the tent.



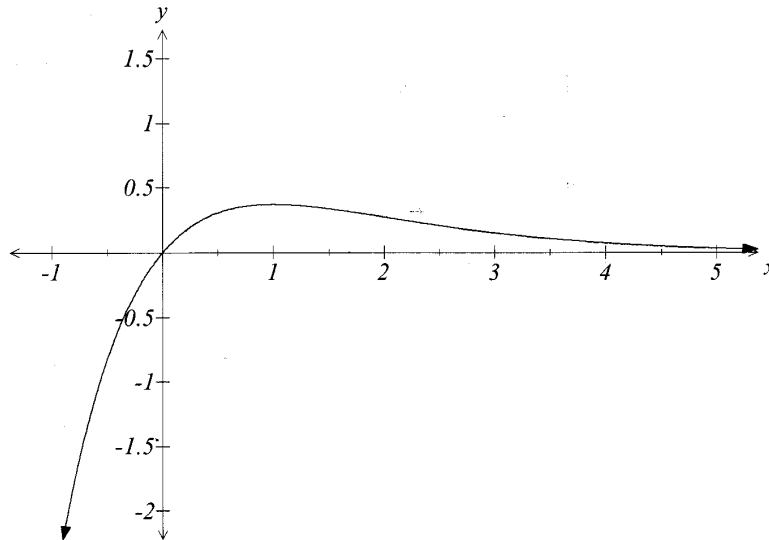
- (b) The region bounded by $y = 4 - x^2$, $x = 2$ and $y = 4$ is rotated about the line $x = 4$. Draw a sketch of the above region. 2
- (i) Using the method of cylindrical shells, show that the volume of a cylindrical shell of thickness δx is 2
$$\pi x^2(8 - 2x)\delta x.$$
- (ii) Find the volume of the solid generated. 2

Question 3 continues over the page

Question 3 Continued

Marks

(c) The graph of $y = xe^{-x}$ is sketched below:



On separate axes, sketch the following curves. Indicate clearly any turning points, asymptotes, and intercepts with the coordinate axes.

(i) $y = x^2 e^{-2x}$ 2

(ii) $y = \frac{1}{x^2 e^{-2x}}$ 2

(iii) $y = \log_e(xe^{-x})$ 2

(iv) $y = e^x e^{-x}$ 2

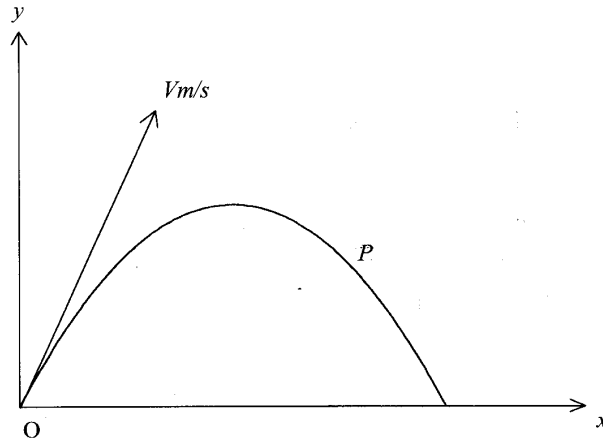
QUESTION 4 (15 MARKS) Start question in new booklet

- Marks**
- (a) For each of the following, $z = x + yi$, where x and y are real:
- (i) Give the Cartesian equation which describes the locus: $2|z| = z + \bar{z} + 4$. 2
- (ii) Draw a neat sketch of the locus specified by: $\text{Arg}(z - i) = \frac{\pi}{4}$. 2
- (iii) On the Argand diagram carefully shade in the region specified by: 3
 $1 \leq |z - 2| \leq 2$ and $\frac{\pi}{6} \leq \text{Arg } z \leq \frac{\pi}{2}$.
- (c) Consider the identity: $\cos 5\theta = 16\cos^5 \theta - 20\cos^3 \theta + 5\cos \theta$. DO NOT PROVE THIS.
- (i) Solve the equation $16x^5 - 20x^3 + 5x - 1 = 0$ in terms of $\cos \theta$, 2
where $0 < \theta < 2\pi$.
- (ii) Hence, or otherwise, show that $\cos \frac{2\pi}{5} + \cos \frac{4\pi}{5} = -\frac{1}{2}$. 2
- (iii) Hence, or otherwise, show that $\cos \frac{2\pi}{5} \cos \frac{4\pi}{5} = -\frac{1}{4}$. 2
- (iv) Hence, deduce the exact values of $\cos \frac{2\pi}{5}$ and $\cos \frac{4\pi}{5}$. 2

QUESTION 5 (15 MARKS) Start question in new booklet

Marks

A particle P is projected from a point O at ground level with a speed V m/s at an angle of elevation α° as shown below:



Given that the equations of motion for the particle P at time t seconds is given by (air resistance is neglected):

$$x = Vt \cos \alpha \quad \text{and} \quad y = -\frac{gt^2}{2} + Vt \sin \alpha \quad \text{DO NOT PROVE THESE EQUATIONS.}$$

- (a) If the highest point of the trajectory of the particle P has coordinates (C, H) ,
- (i) Show that the angle of projection is $\tan^{-1} \frac{2H}{C}$. 4
- (ii) Show that the speed of projection is given by $V^2 = \frac{g}{2H} (4H^2 + C^2)$. 3

where g is the acceleration due to gravity and measured in ms^{-2} .

- (b) At the same time that particle P is projected, a second particle Q is projected horizontally with speed U m/s from a point at height h metres vertically above O , so that the particles move in the same vertical plane.
- (i) Show that if the particles collide, then $V > U$. 3
- (ii) Find the time at which collision takes place, in terms of h , V and U . 2
- (iii) Show that, if the particles collide at ground level, then 3

$$V^2 = U^2 + \frac{1}{2}gh.$$

where g is the acceleration due to gravity measured in ms^{-2} .

QUESTION 6 (15 MARKS) Start question in new booklet

- | | Marks |
|--|--------------|
| (a) A party of 12 people is divided at random into 3 groups of 4 people in each. | 2 |
| (i) In how many ways can the 12 people be divided into the 3 groups? | 1 |
| (ii) Three particular people John, Tanya and Sophie wish to be in the same group. What is the probability that these three people are in the same group? | 1 |
| (b) (i) Prove $\int_0^a f(x) dx = \int_0^a f(a-x) dx$. | 2 |
| (ii) Hence, evaluate for $m > 0$, $\int_0^{\frac{\pi}{2}} \frac{\cos^m x}{\cos^m x + \sin^m x} dx$. | 3 |
| (c) The equilateral triangle ABC lie on a circle as shown. The point P lies on the arc BC , and CP is extended to D such that $PB = PD$. | |

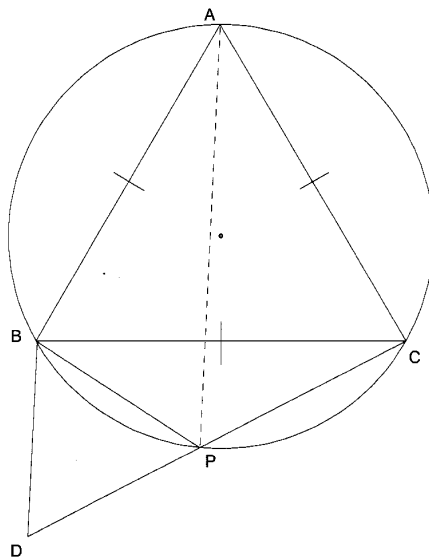


Diagram not to scale

- | | |
|---|---|
| (i) Copy this diagram onto your answer page and label it with all the information.
Prove $\angle ABP = \angle CBD$. | 3 |
| (ii) Hence, prove $\triangle ABP \cong \triangle CBD$. | 2 |
| (iii) Hence, prove $PA = PB + PC$. | 2 |

QUESTION 7 (15 MARKS) Start question in new booklet

Marks

- (a) A polynomial $A(x)$ is divided by $x^2 - a^2$ where $a \neq 0$, and the remainder is $px + q$, so that

$$A(x) = (x^2 - a^2)Q(x) + px + q.$$

- (i) Show that $p = \frac{1}{2a}[A(a) - A(-a)]$ and $q = \frac{1}{2a}[A(a) + A(-a)]$. **3**

- (ii) Find the remainder when $A(x) = x^{2n} - a^{2n}$, is divided by $x^2 - a^2$, where n is a positive integer. **2**

- (b) A particle of mass 10 kg is found to experience a resistive force, in Newtons, of one-ninth of the square of its velocity v , in metres per second, when it moves through the air.

The particle is projected vertically upwards from a point O with a velocity of $30\sqrt{3}$ m/s and the point A , vertically above O , is the highest point reached by the particle before it starts to fall to the ground again.

Assuming the value of $g = 10 \text{ ms}^{-2}$.

- (i) Explain why $\ddot{x} = -10 - \frac{1}{90}v^2$. **1**

- (ii) Find the time the particle takes to reach A from O . **2**

- (c) A vehicle is travelling along a horizontal straight road with a speed of 42 ms^{-1} . The engine is stopped as it passes a point on the road marked O and then the car is allowed to come to rest at a point B . The frictional resistance force is $\frac{1}{7}$ of the weight of the car and the air resistive force is $\frac{v}{14}$ per unit mass, where v is the speed of the car.

- (i) If x is the distance travelled in metres, show why $\ddot{x} = -\left(\frac{v+2g}{14}\right)$, **1**
where g is the acceleration due to gravity, in ms^{-2} .

- (ii) Find the time taken (to nearest second) for the car to come to rest once the engine is stopped. Take $g = 10 \text{ ms}^{-2}$. **3**

- (iii) Find the distance travelled (to the nearest metre) for the car to come to rest once the engine is stopped. Take $g = 10 \text{ ms}^{-2}$. **3**

QUESTION 8 (15 MARKS) Start question in new booklet

(a) Find $\int \frac{1}{x(\ln x)^2} dx$. Marks
2

(b) (i) If k is an integer where $k \geq 3$ and $k^2 > (k-1)(k+1)$, show that 1

$$\frac{1}{k^3} < \frac{1}{(k-1)k(k+1)}.$$

(ii) Given that $S_n = \frac{1}{3^3} + \frac{1}{4^3} + \frac{1}{5^3} + \dots + \frac{1}{n^3} = \sum_{k=3}^n \frac{1}{k^3}$, use partial fractions 5

in part (i) or otherwise, to prove that $S_n < 12$.

(c) (i) Show that: 2

$$(1+x)^{2n} + (1-x)^{2n} = 2 \sum_{r=0}^n {}^{2n}C_{2r} x^{2r} \text{ for } n = 1, 2, 3, \dots$$

(ii) An alphabet consists only of three letters A, B and C.

(α) Explain why the number of words consisting of five letters containing exactly 2 A's is given by ${}^5C_2 \times 2^3$. 2

(β) Use the result in (i), or otherwise, to show that the number of words consisting of $2n$ letters having zero or an even number of A's, is given by: 3

$$\frac{1}{2}(3^{2n} + 1).$$

END OF EXAMINATION