Oakhill

COLLEGE

Mathematics Extension 2

2009
Year 12 Trial Higher School Certificate Examination

Assessment Value: 40 %

General Instructions

- Reading time – 5 minutes
- Writing time – 3 hours
- Write using black or blue pen
- Board approved calculators may be used
- A table of standard integrals is provided at the back of this paper
- All necessary working should be shown in every question

Total Marks – 120

- All questions should be attempted
- All questions are of equal value

Examiner: Mrs P. O'Reilly
Assessor: Mr D. Findlay
Reviewer: Mr T. Attard
Marker: Mrs E. Batfay

<u>QUESTION 1</u> (15 MARKS) Start question in new booklet

Marks

(a) (i) Find real numbers a and b such that: **2**

$$\frac{5x^2-3x+1}{\left(x^2+1\right)\left(x-2\right)}=\frac{ax+1}{x^2+1}+\frac{b}{x-2}.$$

(ii) Hence, find $\int\frac{5x^2-3x+1}{\left(x^2+1\right)\left(x-2\right)}\,dx$. **2**

(b) (i) Show that $\tan^3\theta=\sec^2\theta\,\tan\theta-\tan\theta$. **1**

(ii) Hence, find $\int\tan^3\theta\,d\theta$. **2**

(c) Find $\int\frac{dx}{\sqrt{x^2-4x+2}}$. **3**

(d) (i) Let $I_n=\int_1^e x(\ln x)^n\,dx,\quad n=0,1,2,3,\ldots\ldots.$

Using integration by parts, show that $I_n=\frac{e^2}{2}-\frac{n}{2}I_{n-1}\quad n=1,2,3,\ldots..$ **2**

(ii) The area bounded by the curve $y=\sqrt{x}\left(\ln x\right)^2$, $x\geq 1$, the x-axis and the line $x=e$ is rotated about the x-axis through 2π radians.

Find the exact volume of the solid of revolution formed. **3**

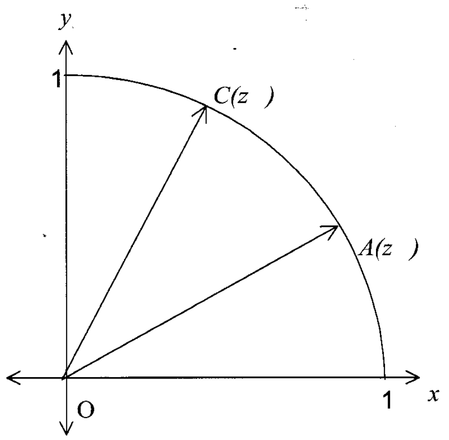
QUESTION 2 (15 MARKS) Start question in new booklet

Marks

(a) Express $z = \dfrac{7+4i}{3-2i}$ in the form $a + ib$, where a and b are real. **2**

(b) On an Argand diagram sketch the locus of the points representing the complex number z where $|z-3-i| = \sqrt{10}$.
Hence, find the greatest value of $|z|$ subject to this condition. **3**

(c)



In the Argand diagram above, the two points A and C lie on the circumference of the circle with centre the origin of radius 1. They represent the complex numbers z_1 and z_2 respectively.

(i) Copy the diagram into your answer sheet. Mark on your diagram the position of the point B that represents the complex number $z_1 + z_2$. **1**

(ii) Explain why AC is perpendicular to OB. **1**

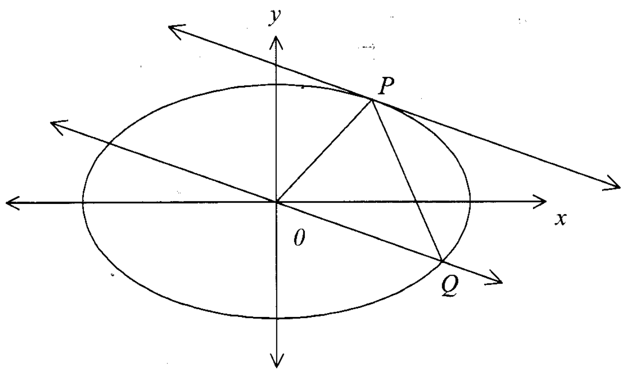
Question 2 continues over the page

Question 2 Continued

Marks

(d) In the diagram below, $P(a\cos\theta,\ b\sin\theta)$ is a point on the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, where P lies in the first quadrant.

A straight line through the origin parallel to the tangent at P meets the ellipse at the point Q, where P and Q both lie on the same side of the y-axis.



(i) Prove that the equation of the line OQ is $x\ b\cos\theta + y\ a\sin\theta = 0$. **2**

(ii) Find the coordinates of the point Q given that Q lies in the fourth quadrant. **3**

(iii) Prove that the area of ΔOPQ is independent of the position of P. **3**

QUESTION 3 **(15 MARKS) Start question in new booklet**

Marks

(a) (i) Show that the area of a regular hexagon of sides s units is given by **1**

$$A = \frac{3\sqrt{3}\,s^2}{2} \text{ units}^2.$$

(ii) The diagram below illustrates a dome tent. When erected, the base is a regular hexagon which measures 2 m from one corner to the adjacent corner (internal measurements). **2**

Flexible exterior poles extend between opposite corners in semi-circular arcs to support the tent.

By taking slices parallel to the base, find the volume enclosed by the tent.

(b) The region bounded by $y = 4 - x^2$, $x = 2$ and $y = 4$ is rotated about the line $x = 4$.
Draw a sketch of the above region.

(i) Using the method of cylindrical shells, show that the volume of a cylindrical shell of thickness δx is **2**

$$\pi\, x^2(8 - 2x)\,\delta x.$$

2

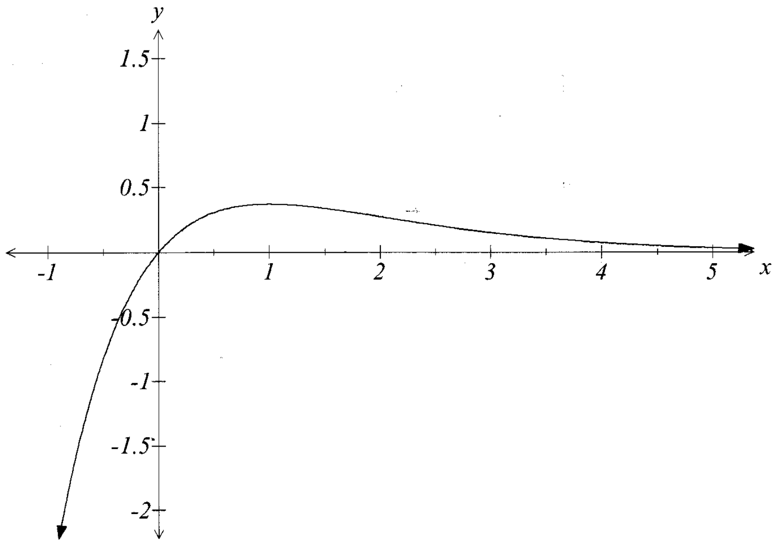
(ii) Find the volume of the solid generated.

Question 3 continues over the page

Question 3 Continued

Marks

(c) The graph of $y = xe^{-x}$ is sketched below:



On separate axes, sketch the following curves. Indicate clearly any turning points, asymptotes, and intercepts with the coordinate axes.

(i) $y = x^2 e^{-2x}$ **2**

(ii) $y = \dfrac{1}{x^2 e^{-2x}}$ **2**

(iii) $y = \log_e\left(xe^{-x}\right)$ **2**

(iv) $y = e^{x e^{-x}}$ **2**

<u>QUESTION 4</u> (15 MARKS) Start question in new booklet

Marks

(a) For each of the following, $z = x + yi$, where x and y are real:

(i) Give the Cartesian equation which describes the locus: $2|z| = z + \overline{z} + 4$. **2**

(ii) Draw a neat sketch of the locus specified by: $A\text{rg}(z - i) = \frac{\pi}{4}$. **2**

(iii) On the Argand diagram carefully shade in the region specified by: **3**

$$1 \le |z - 2| \le 2 \quad \text{and} \quad \frac{\pi}{6} \le Arg\, z \le \frac{\pi}{2}.$$

(c) Consider the identity: $\cos 5\theta = 16\cos^5\theta - 20\cos^3\theta + 5\cos\theta$. DO NOT PROVE THIS.

(i) Solve the equation $16x^5 - 20x^3 + 5x - 1 = 0$ in terms of $\cos\theta$, where $0 < \theta < 2\pi$. **2**

(ii) Hence, or otherwise, show that $\cos\frac{2\pi}{5} + \cos\frac{4\pi}{5} = -\frac{1}{2}$. **2**

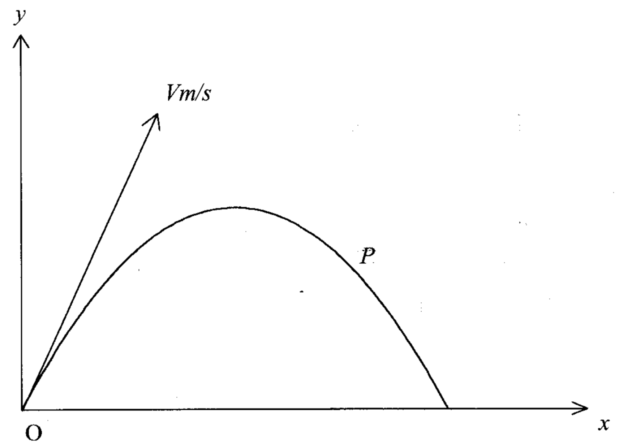
(iii) Hence, or otherwise, show that $\cos\frac{2\pi}{5}\cos\frac{4\pi}{5} = = -\frac{1}{4}$. **2**

(iv) Hence, deduce the exact values of $\cos\frac{2\pi}{5}$ and $\cos\frac{4\pi}{5}$. **2**

<u>QUESTION 5</u> (15 MARKS) Start question in new booklet

Marks

A particle P is projected from a point O at ground level with a speed V m/s at an angle of elevation α ^{o} as shown below:



Given that the equations of motion for the particle P at time t seconds is given by (air resistance is neglected):

$x = Vt\cos\alpha$ and $y = -\dfrac{gt^2}{2} + Vt\sin\alpha$ DO NOT PROVE THESE EQUATIONS.

(a) If the highest point of the trajectory of the particle P has coordinates (C, H),

(i) Show that the angle of projection is $\tan^{-1}\dfrac{2H}{C}$. **4**

(ii) Show that the speed of projection is given by $V^2 = \dfrac{g}{2H}\left(4H^2 + C^2\right)$. **3**

where g is the acceleration due to gravity and measured in ms^{-2}.

(b) At the same time that particle P is projected, a second particle Q is projected horizontally with speed U m/s from a point at height h metres vertically above O, so that the particles move in the same vertical plane.

(i) Show that if the particles collide, then $V > U$. **3**

(ii) Find the time at which collision takes place, in terms of h, V and U. **2**

(iii) Show that, if the particles collide at ground level, then **3**

$$V^2 = U^2 + \frac{1}{2}gh.$$

where g is the acceleration due to gravity measured in ms^{-2}.

<u>QUESTION 6</u> (15 MARKS) Start question in new booklet

Marks

(a) A party of 12 people is divided at random into 3 groups of 4 people in each. **2**

(i) In how many ways can the 12 people be divided into the 3 groups?

(ii) Three particular people John, Tanya and Sophie wish to be in the same group. What is the probability that these three people are in the same group? **1**

(b) (i) Prove $\int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx$. **2**

(ii) Hence, evaluate for $m > 0$, $\int_0^{\frac{\pi}{2}} \frac{\cos^m x}{\cos^m x + \sin^m x}\,dx$. **3**

(c) The equilateral triangle ABC lie on a circle as shown. The point P lies on the arc BC, and CP is extended to D such that $PB = PD$.

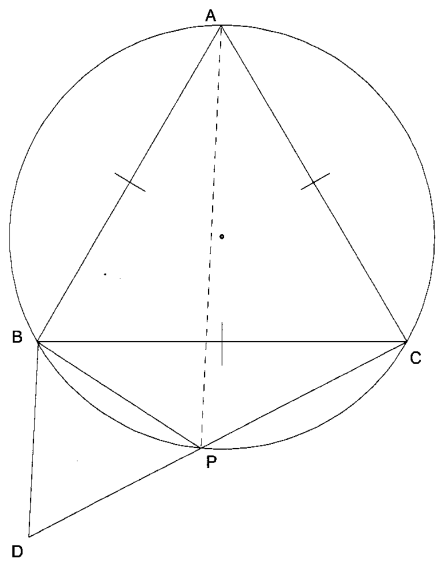


Diagram not to scale

(i) Copy this diagram onto your answer page and label it with all the information.

Prove $\angle ABP = \angle CBD$. **3**

(ii) Hence, prove $\Delta ABP \equiv \Delta CBD$. **2**

(iii) Hence, prove $PA = PB + PC$. **2**

QUESTION 7 (15 MARKS) Start question in new booklet

Marks

(a) A polynomial $A(x)$ is divided by $x^2 - a^2$ where $a \neq 0$, and the remainder is $px + q$, so that

$$A(x) = (x^2 - a^2)Q(x) + px + q.$$

(i) Show that $p = \frac{1}{2a}[A(a) - A(-a)]$ and $q = \frac{1}{2a}[A(a) + A(-a)]$. **3**

(ii) Find the remainder when $A(x) = x^{2n} - a^{2n}$, is divided by $x^2 - a^2$, where n is a positive integer. **2**

(b) A particle of mass 10 kg is found to experience a resistive force, in Newtons, of one-ninth of the square of its velocity v, in metres per second, when it moves through the air.

The particle is projected vertically upwards from a point O with a velocity of $30\sqrt{3}$ m/s and the point A, vertically above O, is the highest point reached by the particle before it starts to fall to the ground again.

Assuming the value of $g = 10$ ms $^{-2}$.

(i) Explain why $\ddot{x} = -10 - \frac{1}{90}v^2$. **1**

(ii) Find the time the particle takes to reach A from O. **2**

(c) A vehicle is travelling along a horizontal straight road with a speed of 42ms^{-1}. The engine is stopped as it passes a point on the road marked O and then the car is allowed to come to rest at a point B. The frictional resistance force is $\frac{1}{7}$ of the weight of the car and the air resistive force is $\frac{v}{14}$ per unit mass, where v is the speed of the car.

(i) If x is the distance travelled in metres, show why $\ddot{x} = -\left(\frac{v+2g}{14}\right)$, where g is the acceleration due to gravity, in ms^{-2}. **1**

(ii) Find the time taken (to nearest second) for the car to come to rest once the engine is stopped. Take g = 10 ms^{-2}. **3**

(iii) Find the distance travelled (to the nearest metre) for the car to come to rest once the engine is stopped. Take g = 10 ms^{-2}. **3**

<u>QUESTION 8</u> (15 MARKS) Start question in new booklet

Marks

(a) Find $\int \frac{1}{x(\ln x)^2}\,dx$. **2**

(b) (i) If k is an integer where $k \geq 3$ and $k^2 > (k-1)(k+1)$, show that **1**

$$\frac{1}{k^3} < \frac{1}{(k-1)k(k+1)}.$$

(ii) Given that $S_n = \frac{1}{3^3} + \frac{1}{4^3} + \frac{1}{5^3} + \ldots\ldots\ldots + \frac{1}{n^3} = \sum_{3}^{n} \frac{1}{k^3}$, use partial fractions **5**

in part (i) or otherwise, to prove that $S_n < 12$.

(c) (i) Show that: **2**

$$(1+x)^{2n} + (1-x)^{2n} = 2\sum_{r=0}^{n} {}^{2n}C_{2r}\, x^{2r} \text{ for } n = 1, 2, 3, \ldots..$$

(ii) An alphabet consists only of three letters A, B and C.

(α) Explain why the number of words consisting of five letters containing exactly 2 A's is given by ${}^5C_2 \times 2^3$. **2**

(β) Use the result in (i), or otherwise, to show that the number of words consisting of $2n$ letters having zero or an even number of A's, is given by: **3**

$$\frac{1}{2}\left(3^{2n}+1\right).$$

END OF EXAMINATION