Student Number: ____________________

Parramatta Marist High School

2020 **YEAR 12 TRIAL 1**

Mathematics Extension 2

General Instructions

- Reading time – 10 minutes
- Working time – 3 hours
- Write using black pen
- Calculators approved by NESA may be used
- A reference sheet is provided at the back of this paper
- Show relevant mathematical reasoning and/or calculations

Total Marks: 100

Section I – 10 marks (pages 2–4)

- Attempt Questions 1–10
- Allow about 15 minutes for this section

Section II – 90 marks (page 5–19)

- Attempt Questions 11–23
- Allow about 2 hours and 45 minutes for this section

Section I

10 marks
Attempt Questions 1–10
Allow about 15 minutes for this section

Use the multiple-choice answer sheet for Questions 1–10.

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1 What is the contrapositive of $P \implies \neg Q$?

A. $Q \implies P$

B. $\neg Q \implies P$

C. $Q \implies \neg P$

D. $\neg Q \implies \neg P$

2 Let $z = 2 - 7i$ and $w = 5 + 3i$.

What is the value of $\overline{z} - 2w$?

A. $-8 - 13i$

B. $-8 + i$

C. $12 - i$

D. $12 + 13i$

3 What is the Cartesian form of $\underset{\sim}{r} = \underset{\sim}{i} \sec\theta + \underset{\sim}{j} \tan\theta$?

A. $x^2 - y^2 = 1$

B. $x^2 + y^2 = 1$

C. $y^2 - x^2 = 1$

D. $x^2 - y^2 = -1$

4 What are the roots of the polynomial $P(x) = x^3 + 3x^2 + 4x + 2$?

A. $1,\ 1 + i,\ 1 - i$

B. $-1,\ 1 + i,\ 1 - i$

C. $1,\ -1 + i,\ -1 - i$

D. $-1,\ -1 + i,\ -1 - i$

5 What is the negation of $\exists x \in \mathbb{Z} : x^2 = -1$?

A. $\exists x \notin \mathbb{Z} : x^2 = -1$

B. $\exists x \in \mathbb{Z} : x^2 \neq -1$

C. $\forall x \in \mathbb{Z} : x^2 = -1$

D. $\forall x \in \mathbb{Z} : x^2 \neq -1$

6 What is the angle between the vectors $\underset{\sim}{u} = \underset{\sim}{i} + \underset{\sim}{k}$ and $\underset{\sim}{v} = \underset{\sim}{j} - \underset{\sim}{k}$?

A. $\frac{\pi}{3}$

B. $\frac{\pi}{2}$

C. $\frac{2\pi}{3}$

D. π

7 Let $z = 1 - i$.

What is z^3 in exponential form?

A. $e^{\frac{3\pi i}{4}}$

B. $e^{-\frac{3\pi i}{4}}$

C. $2^{\frac{3}{2}} e^{\frac{3\pi i}{4}}$

D. $2^{\frac{3}{2}} e^{-\frac{3\pi i}{4}}$

8 The point $(0, 1, -1)$ lies on which line?

A. $\underset{\sim}{r} = \underset{\sim}{i} + \underset{\sim}{j} + \lambda(\underset{\sim}{i} + \underset{\sim}{k})$

B. $\underset{\sim}{r} = \underset{\sim}{i} + \underset{\sim}{j} + \lambda(\underset{\sim}{j} + \underset{\sim}{k})$

C. $\underset{\sim}{r} = \underset{\sim}{i} + \underset{\sim}{k} + \lambda(\underset{\sim}{i} + \underset{\sim}{j})$

D. $\underset{\sim}{r} = \underset{\sim}{i} + \underset{\sim}{k} + \lambda(\underset{\sim}{j} + \underset{\sim}{k})$

9 The probability function $f(x) = \begin{cases} \pi x \sin \pi x, & x \in [0,1] \\ 0, & x \notin [0,1] \end{cases}$.

What is $P(x \leq \frac{1}{2})$?

A. $\frac{1}{\pi^2}$

B. $\frac{1}{\pi}$

C. $\frac{1}{2}$

D. $\frac{\pi}{2}$

10 A particle, initially at rest at the origin, moves with equation of motion $a = 1 + v^2$.

What is the equation of motion for v in terms of x?

A. $v = \tan x$

B. $v = e^x - 1$

C. $v = x + \frac{1}{3}x^3$

D. $v = \sqrt{e^{2x} - 1}$

Section II

90 marks
Attempt Questions 11–23
Allow about 2 hours and 45 minutes for this section

Answer the questions in the spaces provided. Sufficient spaces are provided for typical responses.

Your responses should include relevant mathematical reasoning and/or calculations.

Extra writing space is provided at the back of this booklet. If you use this space, clearly indicate which question you are answering.

Question 11 (3 marks)

Prove that $2^{n+1} + 3^{2n-1}$ is divisible by 7 for $n \in \mathbb{Z}^+$. **3**

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Question 12 (4 marks)

(a) Express $1+i\sqrt{3}$ in exponential form. **2**

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(b) Hence find the two values of $\sqrt{1+i\sqrt{3}}$ in Cartesian form. **2**

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Question 13 (4 marks)

In the box below, shade the region in the complex plane that simultaneously satisfies
$|z| < |z-2-2i|$ and $\frac{\pi}{12} \leq \arg z \leq \frac{5\pi}{12}$.

Question 14 (7 marks)

(a) Let $u_1 = 1$, $u_n = u_{n-1} + n$ for $n \geq 2$. **3**

Prove that $u_n = \frac{1}{2}n(n+1)$ for $n \in \mathbb{Z}^+$.

(b) Hence, prove that $\sum_{k=0}^{n} k^3 = u_n^2$ for $n \in \mathbb{Z}^+$. **4**

Question 15 (15 marks)

(a) Find $\displaystyle\int_0^{\frac{\pi}{2}} \frac{dx}{1+\cos x}$. **3**

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(b) Find $\displaystyle\int \frac{dx}{\sqrt{x^2+2x+2}}$. **4**

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Question 15 continues on page 9

Question 15 (continued)

(c) Find $\int_0^{\frac{\pi}{4}} \tan^3 x \sec^4 x \, dx$. **4**

(d) Find $\int \frac{4\,dx}{(x^2+1)(x-1)}$. **4**

End of Question 15

Question 16 (5 marks)

The displacement x at time t of a particle moving on the x-axis is given by

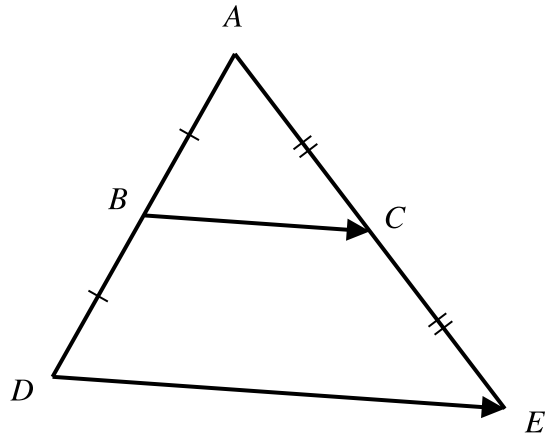
$$x = 3 + \sqrt{3}\sin 3t + \cos 3t.$$

(a) Show that the motion of the particle is simple harmonic. **2**

(b) Find the amplitude and phase of the motion. **3**

Question 17 (4 marks)

In the diagram below, B is the midpoint of AD and C is the midpoint of AE. **4**



Using vectors, prove that $\overrightarrow{BC}$ is half the magnitude of, and parallel to, $\overrightarrow{DE}$.

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Questions 11–17 are worth 42 marks in total.

Question 18 (4 marks)

(a) Prove that $\dfrac{x+y}{2} \geq \sqrt{xy}$ for $x, y \in \mathbb{R}^+$. **2**

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(b) Hence prove that $\dfrac{a}{b} + \dfrac{b}{a} \geq 2$ for $a, b \in \mathbb{R}^+$. **2**

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Question 19 (8 marks)

(a) Consider the sphere given by the Cartesian equation $x^2+y^2+z^2+2x-4z-4=0$. **2**

Show that the vector equation of the sphere is $\left| \underset{\sim}{r} - \begin{bmatrix} -1 \\ 0 \\ 2 \end{bmatrix} \right| = 3$.

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(b) Find the points of intersection between the sphere and the line $\underset{\sim}{r} = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} + \lambda \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$. **3**

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(c) Show that the line $\underset{\sim}{r} = -\underset{\sim}{i} + \underset{\sim}{j} - \underset{\sim}{k} + \mu \underset{\sim}{j}$ is tangent to the sphere. **3**

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Question 20 (7 marks)

(a) Let $I_n = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^n x \, dx$. **3**

Show that for $n \geq 2$, $I_n = \frac{n-1}{n} I_{n-2}$.

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(b) Hence find the volume of revolution for $y = \cos^3 x$ about the x-axis for $x \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$. **4**

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Question 21 (7 marks)

(a) Using De Moivre, show that $\cos 5x = 16\cos^5 x - 20\cos^3 x + 5\cos x$. **3**

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(b) Hence, show that $\cos 18° = \dfrac{\sqrt{10+2\sqrt{5}}}{4}$. **4**

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Question 22 (10 marks)

A projectile is launched from the ground with an initial velocity of u m/s at an angle of θ to the horizontal. The projectile experiences the effect of gravity, and a resistance proportional to its velocity in both the horizontal and vertical directions.

The equations of motion are given by $\underset{\sim}{a} = -0.1\underset{\sim}{v} - 10\underset{\sim}{j}$, where $\underset{\sim}{a}$ is the acceleration vector.

(a) Show that the velocity vector $\underset{\sim}{v} = e^{-0.1t}\left(\underset{\sim}{u} + \underset{\sim}{j}\right) - \underset{\sim}{j}$, where $\underset{\sim}{u} = u\begin{bmatrix}\cos\theta \\ \sin\theta\end{bmatrix}$, satisfies the equations of motion and initial conditions. **3**

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(b) Show that the projectile reaches its peak (maximum height) after $10\ln(u\sin\theta + 1)$ seconds. **3**

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Question 22 continues on page 17

Question 22 (continued)

(c) Let the speed of the projectile at its peak be w m/s. **4**

Show that $\lim_{u \to \infty} w = \cot\theta$.

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End of Question 22

Question 23 (12 marks)

Recall that $x \in \mathbb{Q} \iff \exists p \in \mathbb{Z}, q \in \mathbb{Z}^+ : x = \frac{p}{q}$.

(a) Show that $\forall r \in \mathbb{Z}^+ \exists n \in \mathbb{Z}^+ : 0 < \displaystyle\int_0^1 x^n e^x \, dx < \frac{1}{r}$. **3**

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(b) Prove, using induction, that for all integers $n \geq 0$, $\exists \alpha, \beta \in \mathbb{Z} : \displaystyle\int_0^1 x^n e^x \, dx = \alpha + \beta e$. **4**

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Question 23 continues on page 19

Question 23 (continued)

(c) Let $p \in \mathbb{Z}$, $q \in \mathbb{Z}^+$, $x = \frac{p}{q} \in \mathbb{Q}$.

Show that $\forall \alpha, \beta \in \mathbb{Z} : \left\{ \alpha + \beta x \neq 0 \implies |\alpha + \beta x| \geq \frac{1}{q} \right\}$. **2**

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(d) Hence, prove that $e \notin \mathbb{Q}$. **3**

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End of Question 23

End of Paper

NSW GOVERNMENT

NSW Education Standards Authority

2020 HIGHER SCHOOL CERTIFICATE EXAMINATION

Mathematics Advanced
Mathematics Extension 1
Mathematics Extension 2

REFERENCE SHEET

Measurement

Length

$$l = \frac{\theta}{360} \times 2\pi r$$

Area

$$A = \frac{\theta}{360} \times \pi r^2$$

$$A = \frac{h}{2}(a + b)$$

Surface area

$$A = 2\pi r^2 + 2\pi rh$$

$$A = 4\pi r^2$$

Volume

$$V = \frac{1}{3}Ah$$

$$V = \frac{4}{3}\pi r^3$$

Functions

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

For $ax^3 + bx^2 + cx + d = 0$:

$$\alpha + \beta + \gamma = -\frac{b}{a}$$

$$\alpha\beta + \alpha\gamma + \beta\gamma = \frac{c}{a}$$

and $\alpha\beta\gamma = -\frac{d}{a}$

Relations

$$(x - h)^2 + (y - k)^2 = r^2$$

Financial Mathematics

$$A = P(1 + r)^n$$

Sequences and series

$$T_n = a + (n - 1)d$$

$$S_n = \frac{n}{2}\left[2a + (n - 1)d\right] = \frac{n}{2}(a + l)$$

$$T_n = ar^{n-1}$$

$$S_n = \frac{a(1 - r^n)}{1 - r} = \frac{a(r^n - 1)}{r - 1}, \ r \neq 1$$

$$S = \frac{a}{1 - r}, \ |r| < 1$$

Logarithmic and Exponential Functions

$$\log_a a^x = x = a^{\log_a x}$$

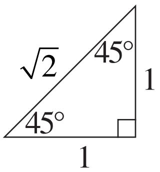
$$\log_a x = \frac{\log_b x}{\log_b a}$$

$$a^x = e^{x \ln a}$$

Trigonometric Functions

$$\sin A = \frac{\text{opp}}{\text{hyp}}, \quad \cos A = \frac{\text{adj}}{\text{hyp}}, \quad \tan A = \frac{\text{opp}}{\text{adj}}$$

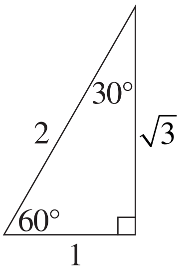
$$A = \frac{1}{2}ab\sin C$$



$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

$$c^2 = a^2 + b^2 - 2ab\cos C$$

$$\cos C = \frac{a^2 + b^2 - c^2}{2ab}$$



$$l = r\theta$$

$$A = \frac{1}{2}r^2\theta$$

Trigonometric identities

$$\sec A = \frac{1}{\cos A}, \ \cos A \neq 0$$

$$\operatorname{cosec} A = \frac{1}{\sin A}, \ \sin A \neq 0$$

$$\cot A = \frac{\cos A}{\sin A}, \ \sin A \neq 0$$

$$\cos^2 x + \sin^2 x = 1$$

Compound angles

$$\sin(A + B) = \sin A\cos B + \cos A\sin B$$

$$\cos(A + B) = \cos A\cos B - \sin A\sin B$$

$$\tan(A + B) = \frac{\tan A + \tan B}{1 - \tan A\tan B}$$

If $t = \tan\frac{A}{2}$ then $\sin A = \frac{2t}{1+t^2}$

$$\cos A = \frac{1-t^2}{1+t^2}$$

$$\tan A = \frac{2t}{1-t^2}$$

$$\cos A\cos B = \frac{1}{2}\left[\cos(A - B) + \cos(A + B)\right]$$

$$\sin A\sin B = \frac{1}{2}\left[\cos(A - B) - \cos(A + B)\right]$$

$$\sin A\cos B = \frac{1}{2}\left[\sin(A + B) + \sin(A - B)\right]$$

$$\cos A\sin B = \frac{1}{2}\left[\sin(A + B) - \sin(A - B)\right]$$

$$\sin^2 nx = \frac{1}{2}(1 - \cos 2nx)$$

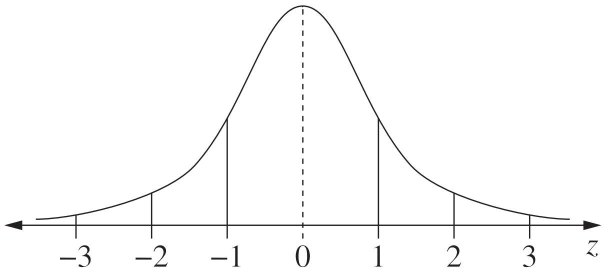
$$\cos^2 nx = \frac{1}{2}(1 + \cos 2nx)$$

Statistical Analysis

$$z = \frac{x - \mu}{\sigma}$$

An outlier is a score
less than $Q_1 - 1.5 \times IQR$
or
more than $Q_3 + 1.5 \times IQR$

Normal distribution



- approximately 68% of scores have z-scores between –1 and 1
- approximately 95% of scores have z-scores between –2 and 2
- approximately 99.7% of scores have z-scores between –3 and 3

$$E(X) = \mu$$

$$\operatorname{Var}(X) = E\left[(X - \mu)^2\right] = E\left(X^2\right) - \mu^2$$

Probability

$$P(A \cap B) = P(A)P(B)$$

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

$$P(A|B) = \frac{P(A \cap B)}{P(B)}, \ P(B) \neq 0$$

Continuous random variables

$$P(X \le r) = \int_a^r f(x)\,dx$$

$$P(a < X < b) = \int_a^b f(x)\,dx$$

Binomial distribution

$$P(X = r) = {}^nC_r p^r (1 - p)^{n-r}$$

$$X \sim \operatorname{Bin}(n, p)$$

$$\Rightarrow \ P(X = x)$$

$$= \binom{n}{x} p^x (1 - p)^{n-x}, \ x = 0, 1, \ldots, n$$

$$E(X) = np$$

$$\operatorname{Var}(X) = np(1 - p)$$

Differential Calculus

Function	Derivative
$y = f(x)^n$	$\dfrac{dy}{dx} = nf'(x)[f(x)]^{n-1}$
$y = uv$	$\dfrac{dy}{dx} = u\dfrac{dv}{dx} + v\dfrac{du}{dx}$
$y = g(u)$ where $u = f(x)$	$\dfrac{dy}{dx} = \dfrac{dy}{du} \times \dfrac{du}{dx}$
$y = \dfrac{u}{v}$	$\dfrac{dy}{dx} = \dfrac{v\dfrac{du}{dx} - u\dfrac{dv}{dx}}{v^2}$
$y = \sin f(x)$	$\dfrac{dy}{dx} = f'(x)\cos f(x)$
$y = \cos f(x)$	$\dfrac{dy}{dx} = -f'(x)\sin f(x)$
$y = \tan f(x)$	$\dfrac{dy}{dx} = f'(x)\sec^2 f(x)$
$y = e^{f(x)}$	$\dfrac{dy}{dx} = f'(x)e^{f(x)}$
$y = \ln f(x)$	$\dfrac{dy}{dx} = \dfrac{f'(x)}{f(x)}$
$y = a^{f(x)}$	$\dfrac{dy}{dx} = (\ln a)f'(x)a^{f(x)}$
$y = \log_a f(x)$	$\dfrac{dy}{dx} = \dfrac{f'(x)}{(\ln a)f(x)}$
$y = \sin^{-1} f(x)$	$\dfrac{dy}{dx} = \dfrac{f'(x)}{\sqrt{1-[f(x)]^2}}$
$y = \cos^{-1} f(x)$	$\dfrac{dy}{dx} = -\dfrac{f'(x)}{\sqrt{1-[f(x)]^2}}$
$y = \tan^{-1} f(x)$	$\dfrac{dy}{dx} = \dfrac{f'(x)}{1+[f(x)]^2}$

Integral Calculus

$$\int f'(x)[f(x)]^n dx = \frac{1}{n+1}[f(x)]^{n+1} + c$$

where $n \neq -1$

$$\int f'(x)\sin f(x)\,dx = -\cos f(x) + c$$

$$\int f'(x)\cos f(x)\,dx = \sin f(x) + c$$

$$\int f'(x)\sec^2 f(x)\,dx = \tan f(x) + c$$

$$\int f'(x)e^{f(x)}dx = e^{f(x)} + c$$

$$\int \frac{f'(x)}{f(x)}dx = \ln|f(x)| + c$$

$$\int f'(x)a^{f(x)}dx = \frac{a^{f(x)}}{\ln a} + c$$

$$\int \frac{f'(x)}{\sqrt{a^2 - [f(x)]^2}}dx = \sin^{-1}\frac{f(x)}{a} + c$$

$$\int \frac{f'(x)}{a^2 + [f(x)]^2}dx = \frac{1}{a}\tan^{-1}\frac{f(x)}{a} + c$$

$$\int u\frac{dv}{dx}dx = uv - \int v\frac{du}{dx}dx$$

$$\int_a^b f(x)\,dx$$

$$\approx \frac{b-a}{2n}\left\{f(a) + f(b) + 2\left[f(x_1) + \cdots + f(x_{n-1})\right]\right\}$$

where $a = x_0$ and $b = x_n$

Combinatorics

$${}^{n}P_{r} = \frac{n!}{(n-r)!}$$

$$\binom{n}{r} = {}^{n}C_{r} = \frac{n!}{r!\,(n-r)!}$$

$$(x+a)^n = x^n + \binom{n}{1}x^{n-1}a + \cdots + \binom{n}{r}x^{n-r}a^r + \cdots + a^n$$

Vectors

$$\left|\underset{\sim}{u}\right| = \left|x\underset{\sim}{i} + y\underset{\sim}{j}\right| = \sqrt{x^2 + y^2}$$

$$\underset{\sim}{u} \cdot \underset{\sim}{v} = \left|\underset{\sim}{u}\right|\left|\underset{\sim}{v}\right|\cos\theta = x_1x_2 + y_1y_2,$$

where $\underset{\sim}{u} = x_1\underset{\sim}{i} + y_1\underset{\sim}{j}$

and $\underset{\sim}{v} = x_2\underset{\sim}{i} + y_2\underset{\sim}{j}$

$$\underset{\sim}{r} = \underset{\sim}{a} + \lambda\underset{\sim}{b}$$

Complex Numbers

$$\begin{aligned} z = a + ib &= r(\cos\theta + i\sin\theta) \\ &= re^{i\theta} \end{aligned}$$

$$\begin{aligned} \left[r(\cos\theta + i\sin\theta)\right]^n &= r^n(\cos n\theta + i\sin n\theta) \\ &= r^n e^{in\theta} \end{aligned}$$

Mechanics

$$\frac{d^2x}{dt^2} = \frac{dv}{dt} = v\frac{dv}{dx} = \frac{d}{dx}\left(\frac{1}{2}v^2\right)$$

$$x = a\cos(nt + \alpha) + c$$

$$x = a\sin(nt + \alpha) + c$$

$$\ddot{x} = -n^2(x - c)$$