Pittwater House

Trial HSC Examination

Mathematics Extension 2

2002

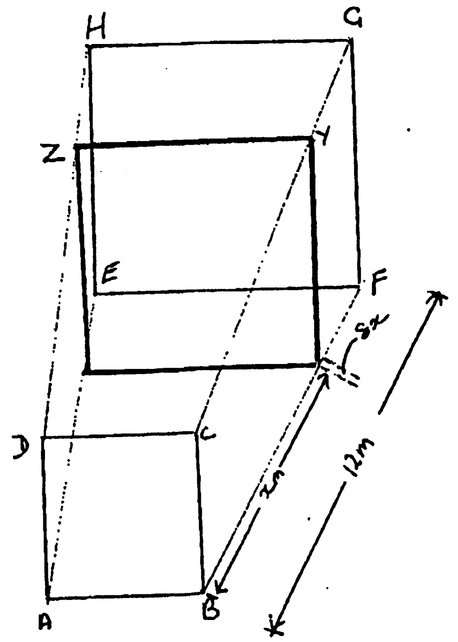
1. (a) Find:
(i) $\int \cot x \operatorname{cosec}^2 x \, dx$
(ii) $\int \frac{\sec^2 x}{3-\tan x} \, dx$
(b) Prove that $\int_{5\frac{1}{2}}^{6\frac{1}{2}} \frac{dx}{\sqrt{(x-5)(7-x)}} = \frac{\pi}{3}$ by using the substitution of $\mu = x - 6$.
(c) (i) Prove that $\int_0^a f(x)dx = \int_0^a f(a-x)dx$
(ii) Hence or otherwise evaluate $\int_0^{\frac{\pi}{2}} \frac{\cos^5 x}{\cos^5 x + \sin^5 x} \, dx$
(d) $\int \frac{1}{4+5\cos x} \, dx$.

2. (a) Evaluate $\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \frac{x^3}{\cos x} \, dx$
(b) Find $\int \sin^3 2x \cos^2 x \, dx$
(c) Find $\int \frac{4x-3}{\sqrt{6+2x-3x^2}} \, dx$
(d) If $I_n = \int_0^{\frac{\pi}{2}} \cos^n x \sin^2 x \, dx$ for $n \geq 0$, show that $I_n = \frac{n-1}{n+2} I_{n-2}$ for $n \geq 2$. Hence or otherwise evaluate $\int_0^{\frac{\pi}{2}} \cos^4 x \sin^2 x \, dx$.

3. (a) (i) Given $z_1 = 1 - i$ and $z_2 = -1 + \sqrt{3}i$, evaluate $|z_1 z_2|$ and $\arg(z_1 z_2)$.
(ii) Find $z_1 z_2$ in Cartesian form, and hence show that $\cos \frac{5\pi}{12} = \frac{\sqrt{3}-1}{2\sqrt{2}}$.
(b) If z is a complex number for which $|z| = 1$, show that:
(i) $1 \leq |z+2| \leq 3$ and
(ii) $-\frac{\pi}{6} \leq \arg(z+2) \leq \frac{\pi}{6}$
(c) (i) Given that $z + \frac{1}{z} = k$, a real number, show that z lies either on the real axis or on the unit circle, centre the origin.
(ii) If z lies on the real axis, show that $|k| \geq 2$. If z lies on the unit circle, show that $|k| \leq 2$.

4. (a) Find integers a and b such that $(x+1)^2$ is a factor of $x^5 + 2x^2 + ax + b$
(b) The equation $z^2 + (1+i)z + k = 0$ has a root $1 - 2i$. Find the other root, and the value of k.
(c) Let α, β and δ be the roots (none of which is zero) of $x^3 + 3px + q = 0$.
(i) Obtain the monic equation whose roots are $\frac{\alpha\beta}{\delta}, \frac{\beta\delta}{\alpha}, \frac{\delta\alpha}{\beta}$
(ii) Deduce that $\delta = \alpha\beta$ if and only if $(3p-q)^2 + q = 0$.

5 (a) The region bounded by the circle $x^2 + y^2 = 4$ and the parabola $y^2 = 3x$ is rotated about the x-axis. By including appropriate diagrams in each case, find the volume of the solid of revolution by using:
(i) circular discs
(ii) cylindrical shells
(b) In the solid shown $ABCD$ and $EFGH$ are squares of side 6m and 10m respectively. $BCGF$ is a trapezium of height 12m. Cross-sections parallel to the ends are squares. Show that at a distance x from the base AB, the area of the cross-section is $(6 + \frac{x}{3})^2$. Hence, by taking slices of thickness δx find the total volume of the solid.

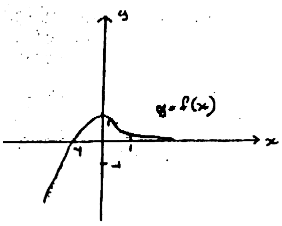


6. **(a)** Show that the ellipse $4x^2 + 9y^2 = 36$ and the hyperbola $4x^2 - y^2 = 4$ intersect at right angles.
(b) You are given that the equation of the normal to the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 (a^2 > b^2)$ at the point $P(x_1, y_1)$ is $a^2 y_1 x - b^2 x_1 y = (a^2 - b^2) x_1 y_1$.
(i) The normal meets the major axis of the ellipse at G. S is a focus of the ellipse. Show that $GS = ePS$, where e is the eccentricity of the ellipse.
(ii) The normal at the point $P(5\cos\theta, 3\sin\theta)$ on $\frac{x^2}{25} + \frac{y^2}{9} = 1$ cuts the major and minor axes of the ellipses at G and H respectively. Show that as P moves on the ellipse, the mid-point of GH describes another ellipse with the same eccentricity at the first.

7. **(a)** The equation of a curve is $x^2 y^2 - x^2 + y^2 = 0$.
(i) Show that the numerical value of y is always less than 1
(ii) Find the equations of the asymptotes
(iii) Show that $\frac{dy}{dx} = \frac{y^3}{x^3}$

(iv) Sketch the curve.
(b) Sketch the following curves on separate axes showing all intercepts and turning points
(i) $y = \cos^2 x$
(ii) $y = x^3 - 4x$
(iii) $y = |x|^3 - 4|x|$

8. (a) For the complex number $w = \sqrt{3} - i$, find w^6 in the form $a + ib$ where a and b are real numbers.
(b) Given that $\arg(z-3) = \arg(z+3) + \frac{\pi}{3}$, show that the locus of points satisfying this equation represents the major arc of a circle where $A(-3,0)$ and $B(3,0)$ are the two fixed points on the arc. State the equation of the locus.
(c) (i) Show the vector diagram representing the complex numbers $z, w, z+w, z-w$
(ii) By geometrical reasons or otherwise, prove that when the complex numbers z and w are such that $|z| = |w|$, then $\frac{z+w}{z-w}$ is purely imaginary.
(d) The graph of $y = f(x)$ is sketched below. There is a stationary point at $(0,1)$.



Use this graph to sketch the following without using calculus, showing essential features:
(i) $y = \frac{1}{f(x)}$
(ii) $y = f\left(\frac{1}{x}\right)$
