Student Name:………………………… Teacher's Name:……………………….. Student Number:……….……….

PRESBYTERIAN LADIES' COLLEGE SYDNEY 1888

2010
TRIAL
HIGHER SCHOOL CERTIFICATE
EXAMINATION

Mathematics Extension 1

General Instructions

- Reading time – 5 minutes
- Working time – 2 hours
- Write using blue or black pen
- Board-approved calculators may be used
- A table of standard integrals is provided at the back of this paper
- All necessary working should be shown in every question

Total Marks – 84

- Attempt questions 1-7
- All questions are of equal value

Question	1	2	3	4	5	6	7	Total	%
Marks	/12	/12	/12	/12	/12	/12	/12	/84	

Question 1 (12 marks) **Start a new sheet of writing paper.** **Marks**

a) Show that $\lim_{x \to 0} \frac{3x}{\tan 2x} = \frac{3}{2}$. **1**

b) $\int_0^{\frac{\pi}{8}} \sin^2 4x \, dx$ **3**

c) Solve $\frac{4x-3}{x} \geq 5$ **3**

d) Find the acute angle between the lines $y = -x - 1$ and $4x + 5y = 2$. Answer to the nearest minute. **3**

e) Using $t = \tan \frac{\theta}{2}$, find the exact value of $\frac{1 - \tan^2 15°}{1 + \tan^2 15°}$ showing all working. **2**

End of Question 1

Question 2 **(12 marks)** **Start a new sheet of writing paper.** **Marks**

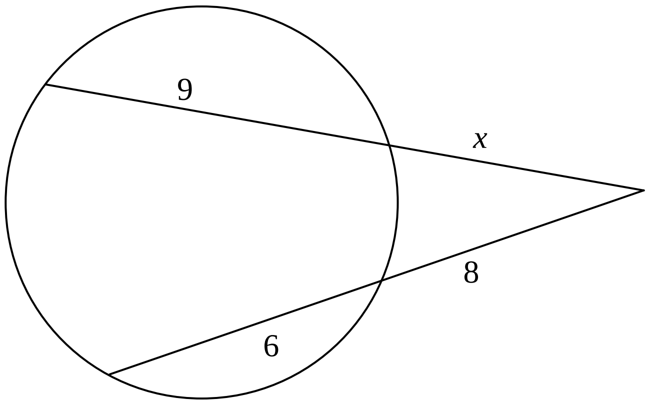
a) Find the coordinates of the point, P, that divides the interval AB externally in the ratio of 1: 4 if A (3, 1) and B (-1, -5). **3**

b) Using the substitution, $u = x^4 + 1$, or otherwise, evaluate $\int_0^1 x^3 e^{x^4+1} \, dx$. **3**

c) Find the constant term in the expansion of $(x^2 - \frac{1}{2x^3})^{10}$ **2**

d) Find the general solution to $2\cos\theta + \sqrt{3} = 0$. Express your answer in terms of π. **2**

e) Find x , giving reasons: **2**



End of Question 2

Question 3 **(12 marks)** **Start a new sheet of writing paper.**

Marks

a) i) Show that $x^3 + 2x - 17 = 0$ has a root between $x=2$ and $x=3$ **1**

ii) Using an approximation of $x = 2\cdot 4$, use one application of Newton's method to find a better approximation for this root. Give your answer to two decimal places. **3**

b) i) Express $\sin x - 2\cos x$ in the form $A\sin(x-\alpha)$ where $0 \le \alpha \le \frac{\pi}{2}$. **2**

ii) Hence, or otherwise, solve $\sin x - 2\cos x = \frac{\sqrt{5}}{2}$ for $0 \le x \le 2\pi$. **2**

Give your answer(s) correct to 2 decimal places.

c) A particle moves on the x axis with velocity v m/s. The particle is initially at rest at x=1 m. Its acceleration is given by $\ddot{x} = 2v$ m/s². Find the velocity and acceleration of the particle at x = 10 metres. **4**

End of Question 3

Question 4 **(12 marks)** **Start a new sheet of writing paper.** **Marks**

a) Given $y = 2\sin^{-1}\frac{x}{3}$

i) State the domain and range of this function. **2**

ii) Sketch the curve $y = 2\sin^{-1}\frac{x}{3}$. **2**

b) Find the exact value of $\int_0^{2\sqrt{3}} \frac{1}{4+x^2}\,dx$ **2**

c) Prove that $\frac{2}{\tan A + \cot A} = \sin 2A$ **2**

d) i) For the binomial expansion of $(4+3x)^{15}$, show that: **2**

$$\frac{T_{k+1}}{T_k} = \frac{16-k}{k} \times \frac{3}{4} \times x\ .$$

ii) Hence, find the greatest coefficient of $(4+3x)^{15}$, leaving your answer in index form. **2**

End of Question 4

Question 5 **(12 marks)** **Start a new sheet of writing paper.** **Marks**

a) The velocity v *m/s* of a particle moving along the x-axis is given by $v^2 = 16x - 4x^2 + 20$.

i) Prove that the motion is simple harmonic. **2**

ii) Find the centre of motion. **1**

iii) Find the distance travelled in one complete oscillation. **1**

b) i) Show that the equation of the normal at $P(2ap, ap^2)$ on the parabola $x^2 = 4ay$ is $x + py - ap^3 - 2ap = 0$ **2**

ii) The normals from $P(2ap, ap^2)$ and $Q(2aq, aq^2)$ on the parabola $x^2 = 4ay$ meet at right angles. Prove that the locus of the points of intersection of these normals is the parabola $x^2 = ay - 3a^2$ **3**

c) Prove by mathematical induction that $\frac{1}{2!} + \frac{2}{3!} + ... + \frac{n}{(n+1)!} = \frac{(n+1)! - 1}{(n+1)!}$ **3**

for all positive integers *n*.

End of Question 5

Question 6 **(12 marks)** **Start a new sheet of writing paper.** **Marks**

a) Newton's law of cooling states that a body cools according to the equation $\frac{dT}{dt} = -k(T-S)$,
where T is the temperature of the body at time t, S is the temperature of the surroundings and k is a constant.

i) Show that $T = S + Ae^{-kt}$ satisfies the equation, where A is a constant. **1**

ii) A metal rod has an initial temperature of $350°C$ and cools to $100°C$ in 10 minutes. The surrounding temperature is $24°C$.

(α) Find the value of A and show that $k = \frac{-1}{10}\log_e\left(\frac{38}{163}\right)$ **2**

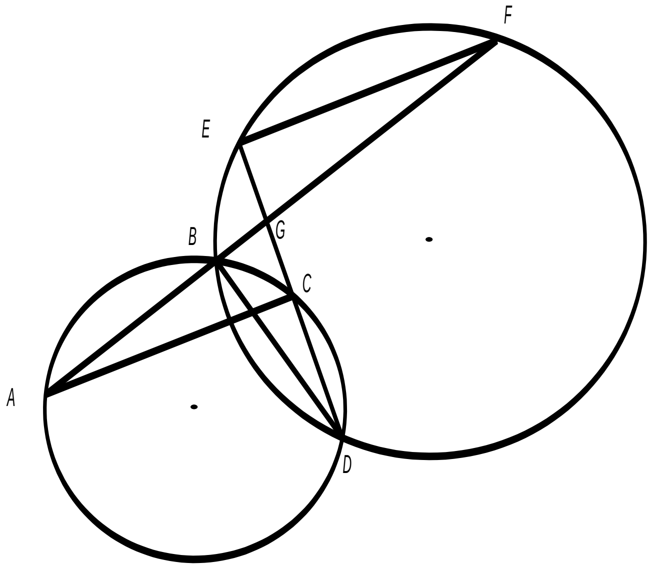
(β) Find how long it will take from the rod to cool to $25°C$. **2**

Question 6 continues on the next page

Question 6 Continued

Marks

b) In the diagram below, ABF and DCE are straight lines.



i) Copy the diagram into your answer booklet.

ii) Prove that AC is parallel to EF. **3**

c) Given that $f(x)=\dfrac{(5-x)(1+x)}{5}$ and $h(x)=\log_e\{f(x)\}$

i) Find the largest domain of $y=h(x)$. **1**

ii) Find the equation of the inverse function $y=h^{-1}(x)$. **2**

iii) Find the domain of the inverse function $y=h^{-1}(x)$. **1**

End of Question 6

Question 7 **(12 marks)** **Start a new sheet of writing paper.**

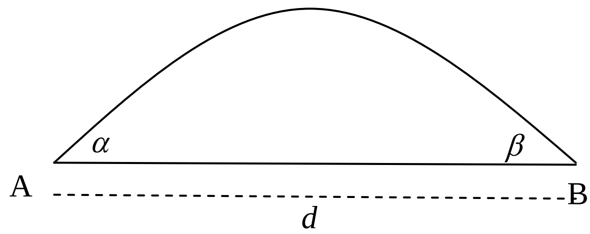
Marks

a) The polynomial $P(x) = x^3 - 2x^2 + ax + b$ has $(x+2)$ and $(x-2)$ as factors, find the

i) values of a and b. **2**

ii) remaining root of $P(x) = x^3 - 2x^2 + ax + b$ **1**

b) A missile is launched from point *A* at an angle α and at a speed *V* towards a target at *B*, *d* metres away. Simultaneously a second missile is launched at speed *W* from *B* at an angle β, to intercept the first. The angles α and β are measured as in the diagram and are related by $\beta = 90° - \alpha$.



The horizontal and vertical displacements of the projectiles from *A* and *B* are given by the following equations (DO NOT PROVE THESE RESULTS):

Missile from *A:*

$x = Vt\cos\alpha$

$y = -\frac{1}{2}gt^2 + Vt\sin\alpha$

Missile from *B*

$x = d - Wt\cos\beta$

$y = -\frac{1}{2}gt^2 + Wt\sin\beta$

i) By equating *y*-components, show that if the missiles are to intersect, then the second missile must have speed $W = V\tan\alpha$. **2**

ii) Show that the time of intersection is $t = \dfrac{d\cos\alpha}{V}$ seconds after launch. **2**

Question 7 continues on the next page

Question 7 Continued

Marks

c) Consider the function $f(x) = \dfrac{\log_e x}{x}$

i) Find the coordinates of the stationary point on the curve $y = f(x)$ and determine its nature. **3**

ii) Hence show that $\pi^e < e^\pi$ **2**

End of Examination

STANDARD INTEGRALS

$$\int x^n \, dx = \frac{1}{n+1} x^{n+1}, \quad n \neq -1; \; x \neq 0, \text{ if } n < 0$$

$$\int \frac{1}{x} \, dx = \ln x, \; x > 0$$

$$\int e^{ax} \, dx = \frac{1}{a} e^{ax}, \; a \neq 0$$

$$\int \cos ax \, dx = \frac{1}{a} \sin ax, \; a \neq 0$$

$$\int \sin ax \, dx = -\frac{1}{a} \cos ax, \; a \neq 0$$

$$\int \sec^2 ax \, dx = \frac{1}{a} \tan ax, \; a \neq 0$$

$$\int \sec ax \tan ax \, dx = \frac{1}{a} \sec ax, \; a \neq 0$$

$$\int \frac{1}{a^2 + x^2} \, dx = \frac{1}{a} \tan^{-1} \frac{x}{a}, \; a \neq 0$$

$$\int \frac{1}{\sqrt{a^2 - x^2}} \, dx = \sin^{-1} \frac{x}{a}, \; a > 0, \; -a < x < a$$

$$\int \frac{1}{\sqrt{x^2 - a^2}} \, dx = \ln\left(x + \sqrt{x^2 - a^2}\right), \; x > a > 0$$

$$\int \frac{1}{\sqrt{x^2 + a^2}} \, dx = \ln\left(x + \sqrt{x^2 + a^2}\right)$$

NOTE : $\ln x = \log_e x, \quad x > 0$

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Course	Ext. 1.	Name of task/exam	Ext. 1 TRIAL EXAM

Question 1 :

a) $\lim_{x\to 0} \frac{3x}{\tan 2x} = \lim_{x\to 0} 3x \frac{x}{\tan 2x}$

$= \lim_{x\to 0} \frac{3x}{2} \frac{2x}{\tan 2x}$

$= \lim_{x\to 0} \frac{3}{2}\left(\frac{2x}{\tan 2x}\right)$

$= \frac{3}{2}(1)$

$= \frac{3}{2}.$

b) $\int_0^{\frac{\pi}{8}} \sin^2 4x \, dx$

we know $\cos 8x = 1 - 2\sin^2 4x$

$\therefore \sin^2 4x = \frac{1}{2} - \frac{1}{2}\cos 8x$

$\therefore \int_0^{\frac{\pi}{8}} \left(\frac{1}{2} - \frac{1}{2}\cos 8x\right) dx$

$= \left[\frac{1}{2}x - \frac{1}{16}\sin 8x\right]_0^{\pi/8}$

$= \left(\frac{1}{2}\left(\frac{\pi}{8}\right) - \frac{1}{16}\sin 8\frac{\pi}{8}\right) - (0 - 0)$

$= \frac{\pi}{16}$

c) $\frac{4x-3}{x} \geq 5$

$x^2\left(\frac{4x-3}{x}\right) \geq 5x^2$

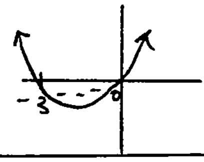
$\therefore x(4x-3) - 5x^2 \geq 0$

$x[4x - 3 - 5x] \geq 0$

$x(-3-x) \geq 0$

$x(x+3) \leq 0$

$\therefore -3 \leq x < 0$



d) $y = -x - 1$ $\quad$ $4x + 5y = 2$

$m = -1$ $\quad$ $m = -\frac{4}{5}$

$\therefore \tan\theta = \left|\frac{m_1 - m_2}{1 + m_1 m_2}\right|$

$= \left|\frac{-1 - -\frac{4}{5}}{1 + (-1)(-\frac{4}{5})}\right|$

$= \left|\frac{-\frac{1}{5}}{\frac{9}{5}}\right|$

$\tan\theta = \frac{1}{9}$

$\therefore \theta = 6° 20'$

e) $t = \tan\frac{\theta}{2}$

$\therefore \cos\theta = \frac{1-t^2}{1+t^2}$

$\therefore \frac{1 - \tan^2 15°}{1 + \tan^2 15°} = \cos 30°$

$\therefore \frac{1 - \tan^2 15°}{1 + \tan^2 15°} = \frac{\sqrt{3}}{2}$

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Course	Ext. 1.	Name of task/exam	TRIAL EXAM

Question 2 :

a) $A\,(\overset{x_1}{3}, \overset{y_1}{1})$ $\quad B\,(\overset{x_2}{-1}, \overset{y_2}{-5})$ $\quad \overset{m}{-1} : \overset{n}{4}$

$$\therefore x = \frac{m x_2 + n x_1}{m+n}, \quad y = \frac{m y_2 + n y_1}{m+n}$$

$$x = \frac{(-1)(-1)+4(3)}{3}, \quad y = \frac{(-1)(-5)+4(1)}{3}$$

$$x = \frac{1+12}{3}, \quad y = \frac{5+4}{3}$$

$$x = \frac{13}{3}, \quad y = 3$$

$$\therefore P\left(\frac{13}{3}, 3\right)$$

b) $\int_0^1 x^3 e^{x^4+1}\,dx$

$$\therefore \int_0^1 \frac{1}{4}\,4x^3 e^{x^4+1}\,dx$$

$u = x^4 + 1$

$\frac{du}{dx} = 4x^3$

$du = 4x^3\,dx$

$u = 1$ when $x = 0$

$u = 2$ when $x = 1$

$$= \int_1^2 \frac{1}{4} e^u\,du$$

$$= \left[\frac{1}{4} e^u\right]_1^2$$

$$= \frac{1}{4}e^2 - \frac{1}{4}e^1$$

$$= \frac{1}{4}e(e-1)$$

c) $\left(x^2 - \frac{1}{2x^3}\right)^{10}$

$$T_{k+1} = {}^{10}C_k\,(x^2)^{10-k}\left(-\frac{1}{2x^3}\right)^k$$

$$T_{k+1} = {}^{10}C_k\,x^{20-2k}\left(-\frac{1}{2}\right)^k x^{-3k}$$

$$= {}^{10}C_k\left(-\frac{1}{2}\right)^k x^{20-5k}$$

for constant term $20-5k=0$

$\therefore k = 4$

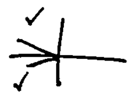
$\therefore$ constant term is ${}^{10}C_4\left(-\frac{1}{2}\right)^4$

$$= 210\left(\frac{1}{16}\right)$$

$$= \frac{105}{8}$$

d) $2\cos\theta + \sqrt{3} = 0$

$$\cos\theta = -\frac{\sqrt{3}}{2}$$



$$\theta = (2k+1)\pi \pm \frac{\pi}{6}, \quad k \text{ integer}$$

e) $(9+x)x = 14 \times 8$ (The product of the intercepts of two intersecting secants to a circle from an external point are equal).

$$9x + x^2 = 112$$

$$x^2 + 9x - 112 = 0$$

$$(x-7)(x+16) = 0$$

$$x = 7, -16$$

but $x > 0$ as length

$\therefore x = 7$

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Question 3:

a) i) $P(x) = x^3 + 2x - 17$

$P(2) = -5$

$P(3) = 16$

since $P(2) < 0$ and $P(3) > 0$ and $P(x)$ is continuous, there exists a root between $x = 2$ and $x = 3$.

ii) $P(2.4) = 1.624$

$P'(x) = 3x^2 + 2$

$P'(2.4) = 19.28$

$\therefore z_2 = z_1 - \dfrac{P(z_1)}{P'(z_1)}$

$= 2.4 - \dfrac{1.624}{19.28}$

$= 2.315\ldots.$

$\therefore$ A better root is $x = 2.32$ (2 dp)

b) $\sin x - 2\cos x \equiv A\sin(x - \alpha)$

$\sin x - 2\cos x \equiv A\sin x\cos\alpha - A\cos x\sin\alpha$

$\therefore 1 = A\cos\alpha$ ①

$2 = A\sin\alpha$ ②

$1 + 4 = A^2(\cos^2\alpha + \sin^2\alpha)$ ①^2 + ②^2

$5 = A^2$

$A = \sqrt{5}$, $A > 0$

$\therefore 2 = \tan\alpha$ ② ÷ ①

$\alpha = 1.107\ldots.$

$\therefore \sin x - 2\cos x \equiv \sqrt{5}\sin(x - 1.107)$

ii) $\sin x - 2\cos x = \dfrac{\sqrt{5}}{2}$

$\therefore \sqrt{5}\sin(x - 1.107) = \dfrac{\sqrt{5}}{2}$

$\sin(x - 1.107) = \dfrac{1}{2}$

$x - 1.107 = \dfrac{\pi}{6}, \dfrac{5\pi}{6}$

$\therefore x = 1.63, 3.73$ (2 dp)

c) $t = 0$ $v = 0$ $x = 1$ m.

$\ddot{x} = 2v$

$v\dfrac{dv}{dx} = 2v$

$\dfrac{dv}{dx} = 2$

$\dfrac{dx}{dv} = \dfrac{1}{2}$

$x = \int \dfrac{1}{2}\,dv$

$x = \dfrac{1}{2}v + c$

$1 = 0 + c$

$\therefore x = \dfrac{1}{2}v + 1$

at $x = 10$

$10 = \dfrac{1}{2}v + 1$

$9 = \dfrac{1}{2}v$

$v = 18$ m/s

$\ddot{x} = 36$ m/s^2

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Question 4 :

a) $y = 2\sin^{-1}\frac{x}{3}$

i) Domain: $-1 \leq \frac{x}{3} \leq 1$

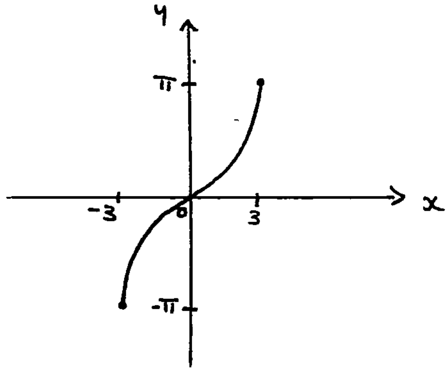
$\therefore \quad -3 \leq x \leq 3$

Range: $-\frac{\pi}{2} \leq \sin^{-1}\frac{x}{3} \leq \frac{\pi}{2}$

$-\pi \leq 2\sin^{-1}\frac{x}{3} \leq \pi$

$-\pi \leq y \leq \pi$

ii)



b) $\int_0^{2\sqrt{3}} \frac{1}{4+x^2}\,dx$

$= \frac{1}{2}\left[\tan^{-1}\frac{x}{2}\right]_0^{2\sqrt{3}}$

$= \frac{1}{2}\left[\tan^{-1}\frac{2\sqrt{3}}{2} - \tan^{-1}\frac{0}{2}\right]$

$\therefore = \frac{1}{2}\left[\tan^{-1}\sqrt{3}\right]$

$= \frac{1}{2}\cdot\frac{\pi}{3}$

$= \frac{\pi}{6}$

c) Prove $\frac{2}{\tan A + \cot A} = \sin 2A$

$LHS = \frac{2}{\frac{\sin A}{\cos A} + \frac{\cos A}{\sin A}}$

$= \frac{2}{\frac{\sin^2 A + \cos^2 A}{\cos A \sin A}}$

$= \frac{2}{\frac{1}{\cos A \sin A}}$

$= 2\cos A \sin A$

$= \sin 2A$

$= RHS$

$\therefore$ proved

d) i) $(4+3x)^{15}$

$T_{k+1} = {}^{15}C_k (4)^{15-k}(3x)^k$

$T_k = {}^{15}C_{k-1}(4)^{15-(k-1)}(3x)^{k-1}$

$\therefore \frac{T_{k+1}}{T_k} = \frac{\frac{15!}{k!(15-k)!}\,4^{15-k}\,3^k\,x^k}{\frac{15!}{(k-1)!(16-k)!}\,4^{16-k}\,3^{k-1}\,x^{k-1}}$

$= \frac{15!}{k!(15-k)!} \times \frac{(k-1)!(16-k)!}{15!} \times 4^{15-k-(16-k)}\,3^{k-(k-1)}\,x$

$= \frac{(16-k)}{k} \times 4^{-1} \times 3^1 \times x^1$

$\frac{T_{k+1}}{T_k} = \frac{16-k}{k} \times \frac{3}{4}x$

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ii) for greatest coefficient

$$\frac{T_{k+1}}{T_k} > 1$$

$$\frac{16-k}{k} \times \frac{3}{4} > 1$$

$$48 - 3k > 4k \quad (\text{as } k > 0)$$

$$48 > 7k$$

$$k < \frac{48}{7}$$

$\therefore k = 6, 5, \ldots$

$\therefore k = 6$ is greatest

$$\therefore T_7 = {}^{15}C_6 (4)^{15-6} (3)^6$$

$$= 5005 \times 4^9 \times 3^6$$

$$= 5005 \times 2^{18} \times 3^6$$

Question 5 :

a) $v^2 = 16x - 4x^2 + 20$

i) $\frac{1}{2}v^2 = 8x - 2x^2 + 10$

$$\frac{d}{dx}\left(\frac{1}{2}v^2\right) = \frac{d}{dx}\left(8x - 2x^2 + 10\right)$$

$$\ddot{x} = 8 - 4x$$

$$\ddot{x} = -4(x-2)$$

which is of the form

$\ddot{x} = -n^2(x-h)$ which is SHM with centre at $x = h$

i.e. $x = 2$.

ii) Centre of motion is $x = 2$

iii) when $v = 0$

$$0 = 16x - 4x^2 + 20$$

$$4x^2 - 16x - 20 = 0$$

$$x^2 - 4x - 5 = 0$$

$$(x+1)(x-5) = 0$$

oscillates between -1 & 5

$\therefore$ distance travelled is 12 m.

b) i) $x^2 = 4ay$

$$y = \frac{x^2}{4a}$$

$$\frac{dy}{dx} = \frac{2x}{4a}$$

at $x = 2ap$

$$m_{tang} = \frac{4ap}{4a} = p.$$

$$\therefore m_{normal} = -\frac{1}{p}$$

$\therefore$ equation normal :

$$y - y_1 = m(x - x_1)$$

$$y - ap^2 = -\frac{1}{p}(x - 2ap)$$

$$py - ap^3 = -x + 2ap$$

$$\therefore x + py - ap^3 - 2ap = 0$$

ii) If normals meet at right angles then

$$-\frac{1}{p} \times -\frac{1}{q} = -1$$

$$\therefore pq = -1$$

PLC Sydney Maths Department

Solutions for exams and assessment tasks

Ver 1

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eqn of normals

$$x + py - ap^3 - 2ap = 0 \quad (1)$$

$$x + qy - aq^3 - 2aq = 0 \quad (2)$$

(1) − (2)

$$y(p-q) - ap^3 + aq^3 - 2ap + 2aq = 0$$

$$y(p-q) - a(p^3 - q^3) - 2a(p-q) = 0$$

$$y(p-q) - a(p-q)(p^2+pq+q^2) - 2a(p-q) = 0$$

since $p \neq q$ can divide by $(p-q)$

$$y - a(p^2+pq+q^2) - 2a = 0$$

$$y = a(p^2+pq+q^2) + 2a$$

$$x = -py + ap^3 + 2ap$$

$$= -p\left[a(p^2+pq+q^2) + 2a\right] + ap^3 + 2ap$$

$$= -ap^3 - p^2qa - apq^2 - 2ap + ap^3 + 2ap$$

$$x = -apq(p+q)$$

∴ pt intersection of Normals:

$$\left(-apq(p+q),\ a(p^2+pq+q^2+2)\right)$$

Locus:

we know $pq = -1$ and

$$x = -apq(p+q) \qquad y = a(p^2+pq+q^2+2)$$

$$x = -a(-1)(p+q)$$

$$x = a(p+q)$$

$$\frac{x}{a} = p+q$$

$$y = a(p^2+pq+q^2) + 2a$$

$$\frac{y-2a}{a} = p^2 + (-1) + q^2$$

$$\frac{y-2a}{a} + 1 = p^2 + q^2$$

$$\frac{y-2a+a}{a} = (p+q)^2 - 2pq$$

$$\frac{y-a}{a} = \left(\frac{x}{a}\right)^2 - 2(-1)$$

$$\frac{y-a}{a} = \frac{x^2}{a^2} + 2$$

$$a(y-a) = x^2 + 2a^2$$

$$\therefore x^2 = ay - a^2 - 2a^2$$

$$x^2 = ay - 3a^2$$

c) Prove

$$\frac{1}{2!} + \frac{2}{3!} + \dots + \frac{n}{(n+1)!} = \frac{(n+1)! - 1}{(n+1)!}$$

for all positive integers n.

Step 1: prove true for $n=1$

$$LHS = \frac{1}{2!} = \frac{1}{2}$$

$$RHS = \frac{(1+1)! - 1}{(1+1)!} = \frac{2! - 1}{2!} = \frac{2-1}{2} = \frac{1}{2}$$

∴ LHS = RHS

∴ true for $n=1$

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Step 2: Assume true for $n=k$

$\therefore \frac{1}{2!} + \frac{2}{3!} + \dots + \frac{k}{(k+1)!} = \frac{(k+1)!-1}{(k+1)!}$

Step 3: Prove true for $n=k+1$

i.e. prove,

$\frac{1}{2!} + \frac{2}{3!} + \dots + \frac{k+1}{(k+2)!} = \frac{(k+2)!-1}{(k+2)!}$

$LHS = \frac{1}{2!} + \frac{2}{3!} + \dots \frac{k}{(k+1)!} + \frac{k+1}{(k+2)!}$

$= \frac{(k+1)!-1}{(k+1)!} + \frac{k+1}{(k+2)!}$

$= \frac{(k+2)[(k+1)!-1] + k+1}{(k+2)!}$

$= \frac{(k+2)! - (k+2) + k+1}{(k+2)!}$

$= \frac{(k+2)! - k - 2 + k + 1}{(k+2)!}$

$= \frac{(k+2)!-1}{(k+2)!}$

$= RHS$

$\therefore$ by the principle of mathematical induction, proved for all positive integers, n.

Question 6:

a) $\frac{dT}{dt} = -k(T-S)$

i) $T = S + Ae^{-kt}$

$T - S = Ae^{-kt}$ ..... ①

also

$\frac{dT}{dt} = Ae^{-kt}(-k)$

$= (T-S)(-k)$ from ①

$\frac{dT}{dt} = -k(T-S)$

ii) $t=0$ $T = 350°C$

$t = 10$ mins $T = 100°C$

$S = 24°C$

(α) $T = 24 + Ae^{-kt}$

$350 = 24 + Ae^{0}$

$A = 326$

$\therefore T = 24 + 326e^{-kt}$

$100 = 24 + 326e^{-k \times 10}$

$76 = 326e^{-10k}$

$\frac{76}{326} = e^{-10k}$

$\ln\left(\frac{76}{326}\right) = -10k$

$k = -\frac{1}{10}\ln\left(\frac{76}{326}\right)$

$k = -\frac{1}{10}\ln\left(\frac{38}{163}\right)$

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(f) $T = 24 + 326e^{-kt}$

$25 = 24 + 326e^{-kt}$

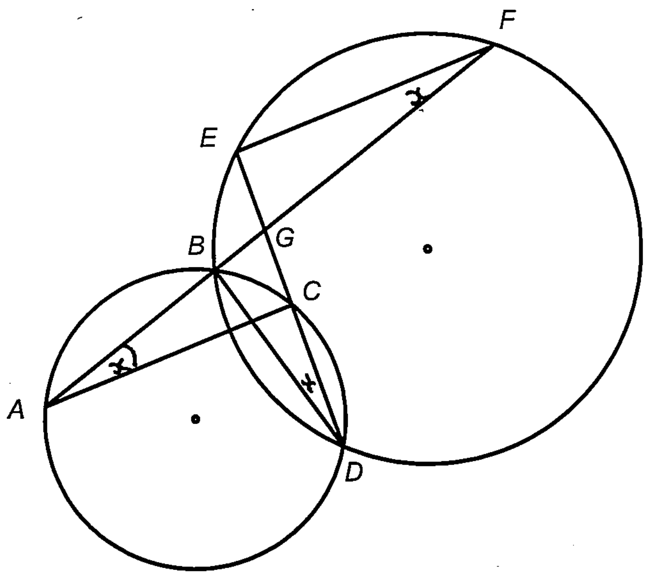
$\frac{1}{326} = e^{-kt}$

$\ln\left(\frac{1}{326}\right) = -kt$

$t = -\frac{1}{k}\ln\left(\frac{1}{326}\right)$

$t = 39 \cdot 7$ mins

b) i)



ii) Let $\angle BAC = x$

$\therefore \angle BDC = x$ (angles standing on same arc BC are equal)

$\therefore \angle BFE = x$ (angles standing on same arc BE equal)

$\therefore \angle BAC = \angle EFG$ and they are alternate angles.

$\therefore AC \parallel EF$

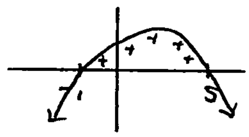
c) $f(x) = \frac{(5-x)(1+x)}{5}$

$h(x) = \ln\{f(x)\}$

i) D: $f(x) > 0$

$\therefore \frac{(5-x)(1+x)}{5} > 0$

$(5-x)(1+x) > 0$



$\therefore -1 < x < 5$

ii) f: $y = \ln\left[\frac{(5-x)(1+x)}{5}\right]$

f^{-1}: $x = \ln\left[\frac{(5-y)(1+y)}{5}\right]$

$e^x = \frac{(5-y)(1+y)}{5}$

$5e^x = (5-y)(1+y)$

$5e^x = 5 + 4y - y^2$

$-5e^x = y^2 - 4y - 5$

$-5e^x = y^2 - 4y + 4 - 9$

$9 - 5e^x = (y-2)^2$

$y - 2 = \pm\sqrt{9-5e^x}$

$y = 2 \pm \sqrt{9-5e^x}$

for function, either of these both not both.

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iii) Domain: $9 - 5e^x \geq 0$

$$5e^x \leq 9$$
$$e^x \leq \frac{9}{5}$$
$$x \ln_e e \leq \ln \frac{9}{5}$$
$$x \leq \ln \frac{9}{5}$$

Question 7:

a) i) $P(x) = x^3 - 2x^2 + ax + b$

$P(-2) = 0$ and $P(2) = 0$

$\therefore -8 - 8 - 2a + b = 0$ ①

$8 - 8 + 2a + b = 0$ ②

$-2a + b = 16$ ①

$2a + b = 0$ ②

$2b = 16$ ① + ②

$b = 8$

$a = -4$

ii) $\therefore P(x) = x^3 - 2x^2 - 4x + 8$

$= (x+2)(x-2)(x-\alpha)$

$$\begin{array}{r} x - 2 \\ x^2 - 4 \,\big)\, x^3 - 2x^2 - 4x + 8 \\ x^3 \quad\quad - 4x \\ \hline -2x^2 \quad + 8 \\ -2x^2 \quad + 8 \\ \hline \end{array}$$

$\therefore$ other root is $x = 2$.

$\therefore$ roots are $2, 2, -2$

b) i) $y = -\frac{1}{2} g t^2 + V t \sin \alpha$ ①

$y = -\frac{1}{2} g t^2 + W t \sin \beta$

$-\frac{1}{2} g t^2 + V t \sin \alpha = -\frac{1}{2} g t^2 + W t \sin \beta$

$V t \sin \alpha = W t \sin \beta$

$V \sin \alpha = W \sin \beta$

$\beta = 90 - \alpha$

$\therefore V \sin \alpha = W \sin (90 - \alpha)$

$V \sin \alpha = W \cos \alpha$

$\therefore W = V \frac{\sin \alpha}{\cos \alpha}$

$W = V \tan \alpha$

ii) $V t \cos \alpha = d - W t \cos \beta$

$V t \cos \alpha + W t \cos \beta = d$

$t (V \cos \alpha + W \cos \beta) = d$

$$t = \frac{d}{(V \cos \alpha + V \tan \alpha \cos \beta)}$$

$$= \frac{d}{(V \cos \alpha + V \tan \alpha \cos (90 - \alpha))}$$

$$= \frac{d}{(V \cos \alpha + V \frac{\sin \alpha}{\cos \alpha} \cdot \sin \alpha)}$$

$$t = \frac{d}{\frac{V \cos^2 \alpha + V \sin^2 \alpha}{\cos \alpha}}$$

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$$t = \frac{d\cos\alpha}{v(\cos^2\alpha + \sin^2\alpha)}$$

$$t = \frac{d\cos\alpha}{v}$$

c) $f(x) = \frac{\ln x}{x}$

i) $f'(x) = \frac{x\left(\frac{1}{x}\right) - \ln x\,(1)}{x^2}$

$= \frac{1 - \ln x}{x^2}$

for stat. pts $f'(x) = 0$

$\therefore \frac{1 - \ln x}{x^2} = 0$

$1 - \ln x = 0$

$\ln x = 1$

$\therefore e = x$

when $x = e$ $\quad y = \frac{\ln e}{e}$

$= \frac{1}{e}$

$\therefore$ stat. pt $\left(e, \frac{1}{e}\right)$

$f''(x) = \frac{x^2\left(-\frac{1}{x}\right) - (1 - \ln x)2x}{x^4}$

$= \frac{-x - 2x + 2x\ln x}{x^4}$

$\therefore f''(x) = -\frac{3x + 2x\ln x}{x^4}$

when $x = e$

$f''(x) = -\frac{3e + 2e\ln e}{e^4}$

$= -\frac{3e + 2e}{e^4}$

$= -\frac{1}{e^3}$

< 0

$\therefore$ max at $\left(e, \frac{1}{e}\right)$.

ii) Since max value is $\frac{1}{e}$

then $\frac{\ln x}{x} < \frac{1}{e}$

at $x = \pi$

$\frac{\ln \pi}{\pi} < \frac{1}{e}$

$e \ln_e \pi < \pi$

$\ln \pi^e < \pi$

take e of both sides

$e^{\ln \pi^e} < e^{\pi}$

$\therefore \pi^e < e^{\pi}$