Polynomials in Leaving Certificate Papers

1920. The roots of the equation $x^4+px^3+qx^2+rx+s=0$ are $\alpha,\beta,\gamma,\delta$. Prove that $\sum\alpha=-p, \sum\alpha\beta=q, \sum\alpha\beta\gamma=-r, \alpha\beta\gamma\delta=s$. If the sum of two of the roots is zero show that $p^2s+r^2=pqr$.

1940. Factorise $\alpha^3+\beta^3-\gamma^3+3\alpha\beta\gamma$. If α,β,γ are the roots of the cubic equation $x^3-2x^2+2x-2=0$, prove that $\alpha+\beta+\gamma=\beta\gamma+\gamma\alpha+\alpha\beta=\alpha\beta\gamma=2$. Find the value of $\alpha^2+\beta^2+\gamma^2$ and show that only one root of the equation is real. Prove also that $(\alpha+\beta-\alpha)^3+2(\alpha^3+\beta^3-\gamma^3)+12=0$.

1946. The roots of $x^3+qx+r=0$ are α,β,γ. Find in terms of q and r, the coefficients of the quadratic equation of which the roots are $\beta\gamma^2+\gamma\alpha^2+\alpha\beta^2$ and $\beta^2\gamma+\gamma^2\alpha+\alpha^2\beta$. Assuming that, if the two roots of the quadratic are real, then all three roots of the cubic are real, show that the necessary and sufficient condition for the cubic to have three real roots is $4q^3+27r^2\le 0$.

1947.
- Prove that the equation whose roots are $(b-c),(c-a)$, and $(a-b)$ is of the form $x^3-px-q=0$ and express p and q in terms of a,b,c. Prove $\sum(b-c)^2=2p; \sum(b-c)^3=3q; \sum(b-c)^4=2p^2; \sum(b-c)^5=5pq; \sum(b-c)^7=7p^2q$.
- (i) The roots of the equation $x^3+px^2+qx+r=0$ are α,β,γ. Find expressions for p,q,r in terms of α,β,γ. For the equation $x^3-5x+1=0$, find $\sum\alpha,\sum\alpha^2,\sum\alpha^3,\sum\alpha^5$.
(ii) Given that the sum of two of the roots of the equation $x^4+2x^3-4x-4=0$ is zero, find all four roots.

1949. (i) Solve the equation $4x^3-24x^2+23x+18=0$ given that the roots are in arithmetic progression.
(ii) If α,β,γ are the roots of the equation $x^3-7x^2+18x-7=0$, find the value of $(1+\alpha^2)(1+\beta^2)(1+\gamma^2)$.

1950. If α,β,γ are the roots of $x^3-px^2+qx-r=0$, find in terms of $p,q,r:\alpha+\beta+\gamma;\alpha^2+\beta^2+\delta^2;\alpha^3+\beta^3\delta^3$. Hence find a solution of the set of equations $X+Y+Z=-1; X^2+Y^2+Z^2=5; X^3+Y^3+Z^3=-7$.

1951.
- Find the equation of which the roots are (a) the negatives (b) the squares of those of $f(x)\equiv x^4+qx^2+rx+s=0$. If $\alpha,\beta,\gamma,\delta$ are the roots of $f(x)=0$, show that $(1+\alpha^2)(1+\beta^2)(1+\gamma^2)(1+\delta^2)=(1-q+s)^2+r^2$.
- (i) If α is a real root of $x^3+ux+v=0$, prove that the other 2 roots are real if $4u+3\alpha^2\le 0$.
(ii) Find the condition for $x^3-3ax+b=0$ to have 3 real roots $(a>0)$.
- By making the substitution $y=x^2+(a+b)x$, or otherwise, obtain a formal solution of the equation $x(x+a)(x+b)(x+a+b)=k$. Show that all four roots are real if $\frac{1}{4}(a^2+b^2)^2\ge a^2b^2+4k\ge 0$. Find all sets of four numbers (not necessarily integers) which are in arithmetical progression with common difference 1, and are such that their product is $-\frac{504}{625}$.

1952. If $f(x)$ be a polynomial and $f'(x)$ the derived polynomial, show by graphical considerations, or otherwise, that if $f'(a)=0$ and $f'(b)=0$, and $f'(x)\ne 0$ for $a<x<b$, then there is either one root or no root of $f(x)=0$ between a and b. Prove further that if $f(a)=0$ and $f'(a)=0$, then a is a double root (at least) of $f(x)=0$. Find the range of values of k for which $2x^3-9x^2+12x-k=0$ has 3 real roots. Solve completely the equations obtained with the values of k at the extremities of this range.

1953. (i) Prove that if α is an r-fold root of the polynomial equation $f(x)=0$, then it is an $(r-1)$-fold root of $f'(x)=0$. (You may assume $f(x)\equiv(x-\alpha)^p g(x)$ where $g(x)$ is a polynomial).
(ii) If $f(x)$ is a polynomial of degree n, and $\alpha_1,\dots,\alpha_n$ are the roots of $f(x)=0$ (they need not be all real or all different), and $\beta_1,\dots,\beta_{n-1}$ are the roots of $f'(x)=0$, then $(n-1)\sum_{r=1}^{n}\alpha_r=n\sum_{s=1}^{n-1}\beta_s$.
(iii) The only roots of a polynomial equation $f(x)=0$ are $\alpha,\beta\gamma;\alpha$ is r-fold,β is s-fold, γ is t-fold. Show that $f'(x)=0$ has two roots not equal to α,β or γ, and that the sum of these two is $\frac{(s+t)\alpha+(t+r)\beta+(r+s)\gamma}{r+s+t}$. Verify this result for the equation $x^3(x^2-2x+2)^4=0$.

1954. If the n quantities $\alpha,\beta,\gamma,\dots,\lambda$ are all positive, prove that $(\alpha+\beta+\gamma+\dots+\lambda)(\frac{1}{\alpha}+\frac{1}{\beta}+\frac{1}{\gamma}+\dots+\frac{1}{\lambda})\ge n^2$. Prove that if all the roots of the equation $p_0x^n+p_1x^{n-1}+\dots+p_{n-1}x+p_n=0$ are positive, then $p_1p_{n-1}\ge n^2p_0p_n$.

1955. The roots of the equation $x^3+px^2+qx+r=0$ are in geometrical progression, prove that $p^3r=q^3$.