Student's name ________________________ **Student's number** ___________ **Teacher's name** _____________________

PLC PRESBYTERIAN LADIES' COLLEGE SYDNEY 1888

2014
TRIAL
HIGHER SCHOOL CERTIFICATE
EXAMINATION

Mathematics Extension 2

General Instructions

- Reading time - 5 minutes
- Working time - 3 hours
- Write using blue or black pen Black is preferred
- Board-approved calculators may be used
- A table of standard integrals is provided at the back of this paper
- All necessary working should be shown in every question

Total Marks – 100

Section I: Pages 3-6
10 marks

- Attempt questions 1-10, using the answer sheet on page 19.
- Allow about 15 minutes for this section

Section II: Pages 7-16
90 marks

- Attempt questions 11-16, using the booklets provided.
- Allow about 2 hours 45 minutes for this section

Multiple Choice	11	12	13	14	15	16	Total
							%

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Section I

10 marks
Attempt Questions 1–10
Allow about 15 minutes for this section

Use the multiple-choice answer sheet for Questions 1–10.

1. If $z = \left(1 - i\sqrt{3}\right)^{2014}$ what is $Arg\, z$?

(A) $-\dfrac{2014\pi}{3}$

(B) $-\dfrac{2014\pi}{6}$

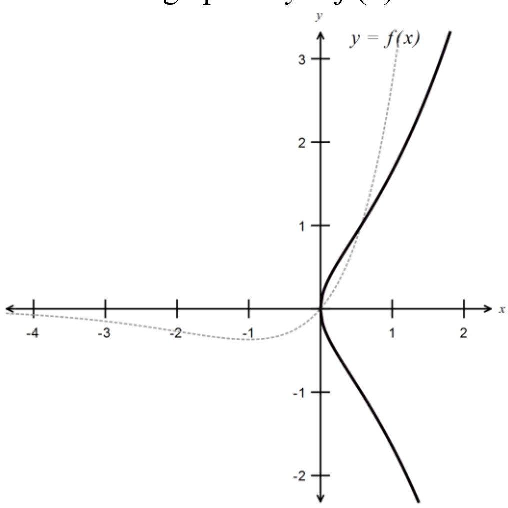
(C) $\dfrac{2014\pi}{3}$

(D) $\dfrac{2014\pi}{6}$

2. What does the equation $x^2 + 2y^2 - 24 = 0$ represent?

(A) Parabola
(B) Hyperbola
(C) Ellipse
(D) None of these

3. Which of the following transformations best describes the graph below? The graph of $y = f(x)$ is shown on the same diagram.



(A) $|y| = f(x)$

(B) $y^2 = f(x)$

(C) $y = |f(x)|$

(D) $y = \left[f(x)\right]^2$

4. Which expression is equal to $\int \frac{dx}{\sqrt{4x^2+1}}$?

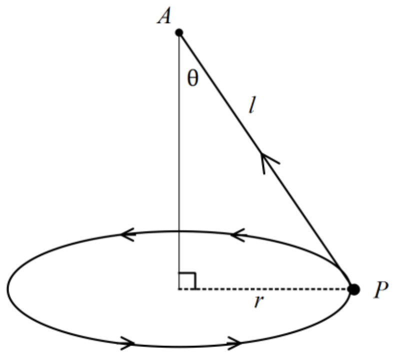
(A) $\sin^{-1} 2x + c$

(B) $\log_e\left(2x+\sqrt{4x^2+1}\right)+c$

(C) $\frac{1}{2}\log_e\left(x+\sqrt{x^2+\frac{1}{4}}\right)+c$

(D) $\frac{1}{4x}\sqrt{4x^2+1}+c$

5. A particle, P, of mass m kilograms, is suspended from a fixed point by a string of length, l metres with acceleration due to gravity, $g\ ms^{-2}$. P is moving with uniform circular motion about a horizontal circle with velocity $\omega\ rads/second$ and radius r. The forces acting on the particle are the gravitational force and the tension force T along the string. **1**



Which of the following expressions are the correct horizontal and vertical components of the force acting on P?

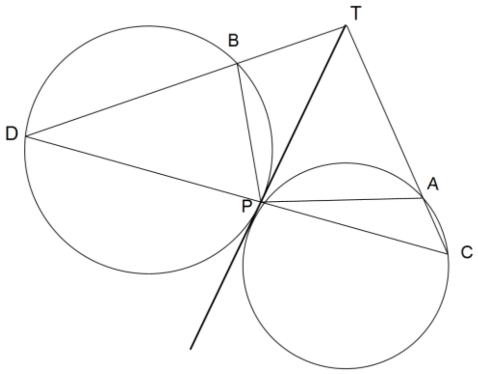
(A) $T\,sin\,\theta = mg$
$T\,cos\,\theta = mr\omega$

(B) $T\,cos\,\theta = mg$
$T\,sin\,\theta = mr\omega$

(C) $T\,sin\,\theta - mg = 0$
$T\,cos\,\theta = mr\omega^2$

(D) $T\,cos\,\theta - mg = 0$
$T\,sin\,\theta = mr\omega^2$

6. If TP is a common tangent to the circles in the diagram below, which line has an error in proving that *ATBP* is a cyclic quadrilateral?



(A) $\angle TPA = \angle TPB$ (common tangent bisects $\angle APB$)

(B) $\angle TPA = \angle PCA$ (angle between the tangent and the chord is equal to the angle in the alternate segment)

(C) $\angle TPB = \angle PDB$ (angle between the tangent and the chord is equal to the angle in the alternate segment)

(D) $\angle DTC = 180 - \angle TDC - \angle TCD$ (angle sum of a triangle)
$\therefore \angle APB + \angle DTC = 180$
Opposite angles in a cyclic quadrilateral are supplementary
$\therefore ATBP$ is a cyclic quadrilateral

7. A particle is moving in a circular path of radius *r*, with a constant angular speed of ω. The normal component of the acceleration is:

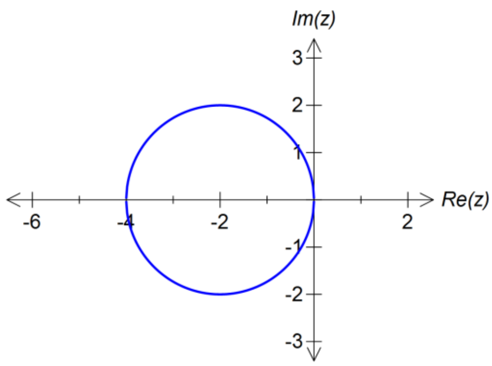
(A) ω

(B) $r\omega$

(C) $r\omega^2$

(D) $(r\omega)^2$

8.



Which one of the following is the equation of the circle in the diagram above?

(A) $(z+2)(\overline{z}+2)=4$

(B) $(z-2)(\overline{z}+2)=4$

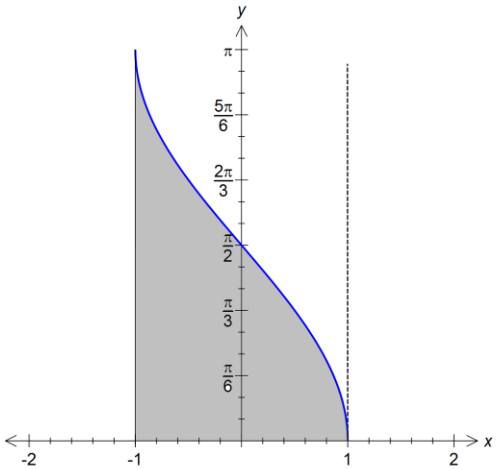
(C) $(z-2)(\overline{z}-2)=4$

(D) $(z+2i)(\overline{z}-2i)=4$

9. The roots of $x^3+5x+3=0$ are α, β and γ. Which one of the following polynomials has roots $\alpha\beta$, $\beta\gamma$ and $\alpha\gamma$?

(A) $x^3-5x^2-9=0$
(B) $x^3+5x^2+9=0$
(C) $x^3-125x-375=0$
(D) $x^3+125x-+375=0$

10. In the diagram, the shaded region is bounded by the x-axis, the line $x=-1$ and the curve $y=\cos^{-1}x$.



Find the volume of the solid formed when this region is rotated about $x=1$.

(A) $\dfrac{3+\pi^2}{2}$

(B) $\dfrac{3}{2}$

(C) $\dfrac{5\pi^2}{2}$

(D) None of the above

Section II

90 marks
Attempt Questions 11–16
Allow about 2 hours and 45 minutes for this section

Answer each question in a SEPARATE writing booklet. Extra writing booklets are available.

In Questions 11–16, your responses should include relevant mathematical reasoning and/or calculations.

Question 11 (15 marks) Use a SEPARATE writing booklet.

a) (i) Show that $tan^3 x = sec^2 x\, tan\, x - tan\, x$. **1**

(ii) Hence evaluate $\int_0^{\frac{\pi}{4}} tan^3 x\, dx$ **2**

b) If $\omega = \dfrac{1 - i\sqrt{3}}{2}$

(i) Show that $\omega^3 = -1$. **2**

(ii) Hence calculate ω^{16} **1**

c) (i) Find $\sqrt{5-12i}$ in $x+iy$ form. **2**

(ii) Hence, or otherwise, solve the equation $z^2 + 4z - 1 + 12i = 0$ **2**

d) Consider the equation $z^3 - z^2 - 2z - 12 = 0$. Given that $z = 2cis\left(\dfrac{2\pi}{3}\right)$ is a root of the equation, factorise fully over the

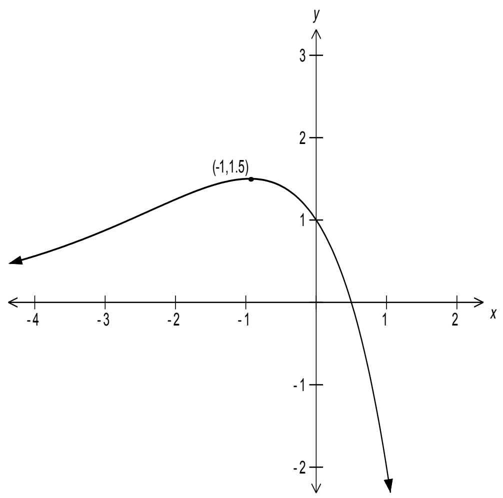
(i) real field **2**

(ii) complex field **1**

Question 11 continued over page

Question 11 continued

e) The following diagram shows the graph of $y = f(x)$.



On your answer sheet, draw separate one-third page sketches of the graphs of the following

(i) $y = -f(x)$ **1**

(ii) $y = \sqrt{f(x)}$ **1**

End of Question 11

Question 12 (15 marks) Use a SEPARATE writing booklet.

a) The equation of the ellipse, E, is $\frac{x^2}{25}+\frac{y^2}{9}=1$.

The point P is on the ellipse with co-ordinates (x_1, y_1).

(i) Find the eccentricity of the ellipse. **1**

(ii) Find the co-ordinates of the foci and the equations of the directrices of the ellipse. **2**

(iii) Show that the equation of the tangent at P is $\frac{x_1 x}{25}+\frac{y_1 y}{9}=1$. **2**

(iv) Let the tangent at P meet a directrix at a point J. Show that $\angle PSJ$ is a right angle where S is the corresponding focus. **3**

b) Consider $f(x)=sin\,x+cos\,x$

(i) Find A and B such that $sin\,x+cos\,x=A\,sin(x+B)$ **2**

(ii) Sketch $f(x)=sin\,x+cos\,x$ for $-2\pi \le x \le 2\pi$. **2**

(iii) Hence, or otherwise, sketch $y=\frac{1}{f(x)}$ for $-2\pi \le x \le 2\pi$. **1**

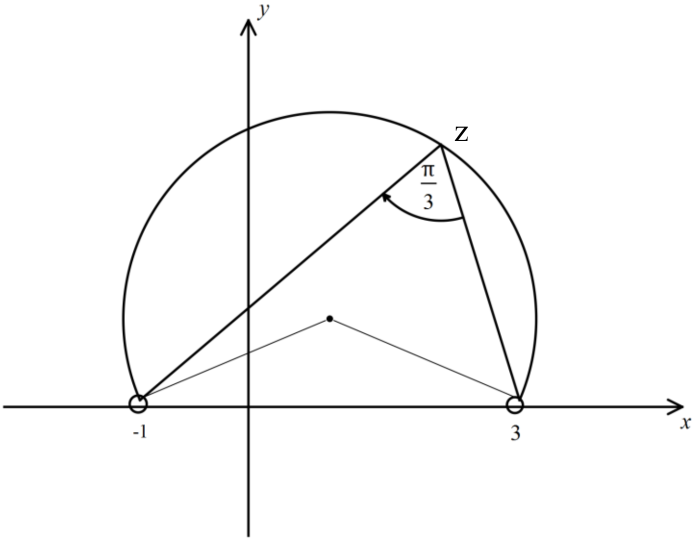
(iv) Sketch $y=\frac{f(x)}{x}$ **2**

End of Question 12

Question 13 (15 marks) Use a SEPARATE writing booklet.

a) The diagram shows the locus of a point z in the complex plane such that **3**

$$arg\left(z-3\right)-arg\left(z+1\right)=\frac{\pi}{3}.$$



This locus is part of a circle. The angle between the lines from -1 to z and from 3 to z is $\frac{\pi}{3}$, as shown.

Find the centre and radius of the circle.

b) Find $\int \frac{x^2+2x}{(x-2)(x^2+4)}dx$ **3**

c) Evaluate $\int_0^{\frac{\pi}{4}} \frac{dx}{cos^2 x+2\,sin^2 x}$ using $t=tan\,x$ **4**

d) (i) Show that $\frac{1+sin\,\theta+i\,cos\,\theta}{1+sin\,\theta-i\,cos\,\theta}=sin\,\theta+i\,cos\,\theta$ **3**

(ii) Hence prove that **2**

$\left(\frac{1+sin\,\theta+i\,cos\,\theta}{1+sin\,\theta-i\,cos\,\theta}\right)^n=cos\left(\frac{n\pi}{2}-n\theta\right)+i\,sin\left(\frac{n\pi}{2}-n\theta\right)$, where n is a positive integer.

End of Question 13

Question 14 (15 marks) Use a SEPARATE writing booklet.

a) Find all the roots of the equation $16x^3 - 4x^2 - 8x + p = 0$ if two of the roots are equal. **3**

b) Prove by Mathematical Induction that **3**
$2^n > n^2$ for all integers $n \geq 5$

c) A particle of mass, *m* kilograms, is initially projected from the ground at an angle of θ, to the horizontal where $\theta = tan^{-1}\left(\frac{3}{4}\right)$ and an initial velocity of $25 m/s$.

The equations of motion for the particle are

$x = 25t\,cos\,\theta$
$y = -\frac{1}{2}gt^2 + 25t\,sin\,\theta$, where g is the acceleration due to gravity.
(DO NOT PROVE THESE RESULTS)

(i) Show that $x = 20t$ and $y = 15t - 5t^2$ if $g = 10 m/s^2$. **2**

(ii) If the particle hits a wall 40 metres from the point of projection

(α) find the height above the ground the particle hits. **1**

(β) show that the velocity of the particle, at the point of impact, is $\sqrt{425}\, m/s$. **2**

(iii) At impact, the particle is instantaneously at rest. It then falls vertically to the ground with a resistance force acting against the vertical motion equal to $0.01mv^2$ Newtons.

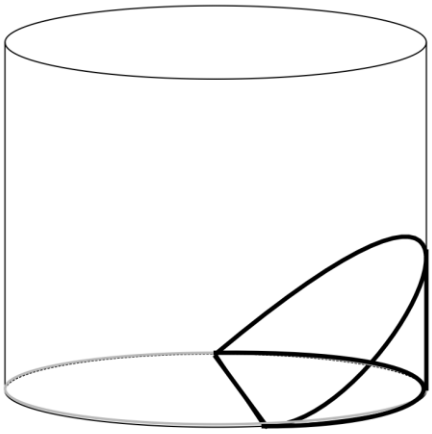
(α) Show that $a = 10 - 0.01v^2$, where a is the acceleration and v is the velocity of the particle. **1**

(β) Find the velocity on returning to the ground. Answer correct to 2 decimal places. **3**

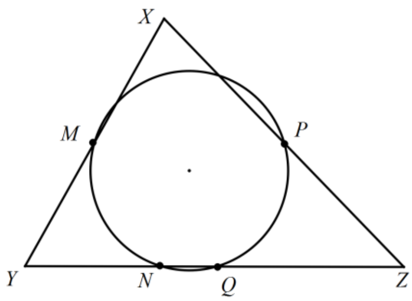
End of Question 14

Question 15 (15 marks) Use a SEPARATE writing booklet.

a) A wedge is cut out of a right circular cylinder of radius 4 centimetres by two planes. One plane is perpendicular to the axis of the cylinder. The other plane intersects the first at an angle of 30^0, along a diameter of the cylinder. Find the volume of the wedge. **3**



b) In the acute-angled triangle XYZ, M is the midpoint of XY, Q is the midpoint of YZ and P is the midpoint of ZX. The circle through M, Q and P also cuts YZ at N as shown in the diagram.



(i) Prove $MPQY$ is a parallelogram. **1**

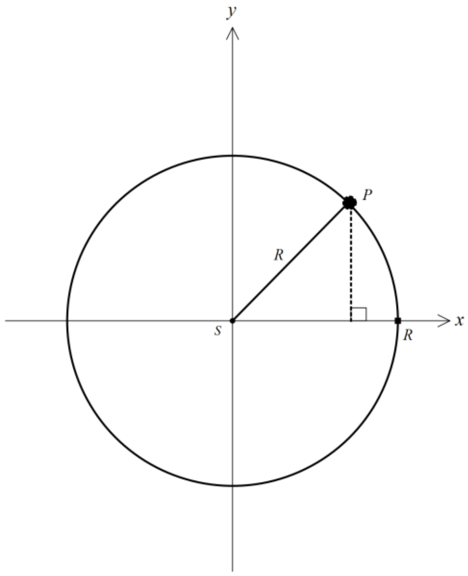
(ii) Prove $\angle MNY = \angle MPQ$. **1**

(iii) Prove that $XN \perp YZ$. **2**

Question 15 continued over page

Question 15 continued

c) A planet P of mass, m kilograms, moves in a circular orbit of radius R metres, around a star, S, in uniform circular motion. The position of the planet at time t seconds is given by the equations $x = R\cos\dfrac{2\pi t}{T}$ and $y = R\sin\dfrac{2\pi t}{T}$, where T is a constant.



(i) Show that $\ddot{x} = \dfrac{-4\pi^2}{T^2}x$ and $\ddot{y} = \dfrac{-4\pi^2}{T^2}y$ **2**

(ii) Show the acceleration of P is $\dfrac{-4\pi^2}{T^2}R$. **1**

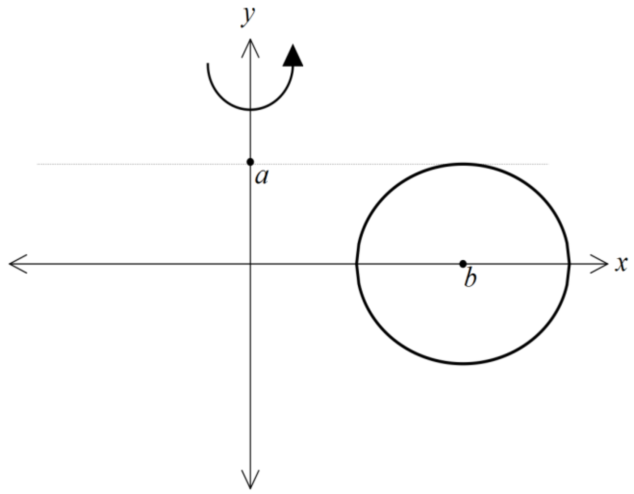
(iii) Find the force exerted by the star, S, on the planet, P. **1**

(iv) It is known that the magnitude of the gravitational force pulling the planet towards the star is given by $F = \dfrac{GMm}{R^2}$, where G is constant and M is the mass of the star, S, in kilograms. Show that the expression for T in terms of R, M and G is $T = 2\pi R\sqrt{\dfrac{R}{GM}}$. **2**

Question 15 continued over page

Question 15 continued

d) A donut shaped solid called a torus is formed by revolving $(x-b)^2 + y^2 = a^2$, $0 < a < b$ about the y-axis. **2**



Express the volume of the torus as a definite integral in x. Do not evaluate this integral.

End of Question 15

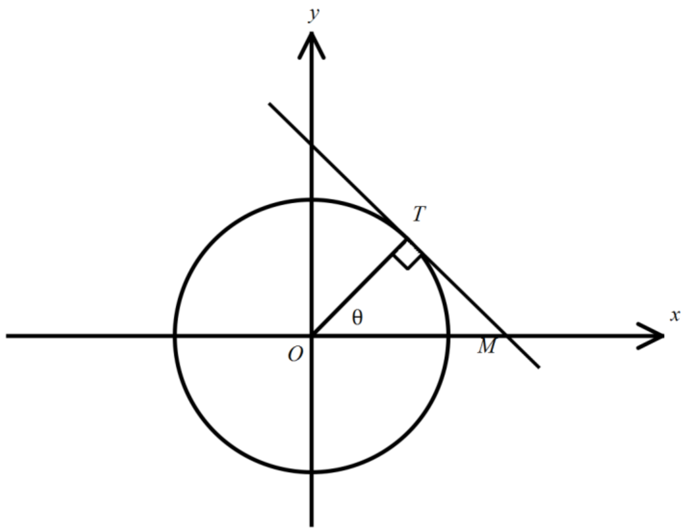
Question 16 (15 marks) Use a SEPARATE writing booklet.

a) If $I_n = \int_0^{\frac{\pi}{2}} \sin^n \theta \, d\theta$ where $n \ge 2$,

(i) Show that $I_n = \frac{(n-1)}{n} I_{n-2}$ **3**

(ii) Hence or otherwise, evaluate $\int_0^2 (4-x^2)^{\frac{5}{2}} dx$. **3**

b) The figure shows the circle $x^2 + y^2 = a^2$.



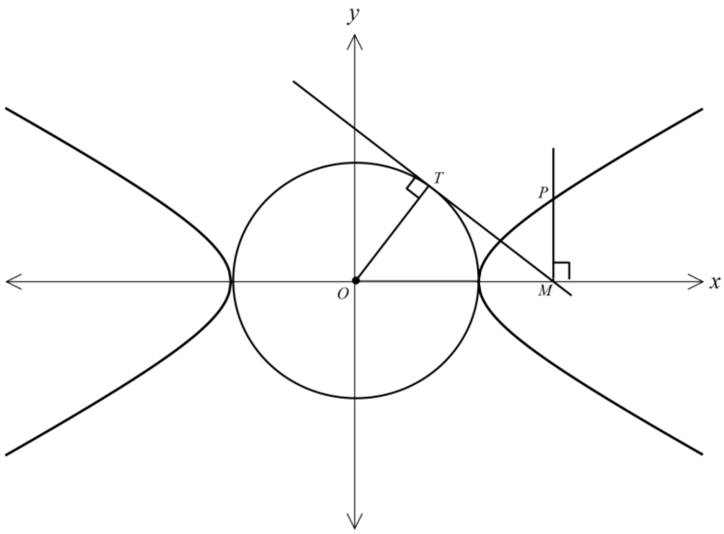
The point T lies on the circle. $\angle TOx = \theta$, where $0 \le \theta \le \frac{\pi}{2}$. The tangent to the circle at T meets the x-axis at M.

(i) Show that the co-ordinates of M are $(a\sec\theta, 0)$. **1**

Question 16 continued over page

Question 16 continued

The hyperbola $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$ and the circle $x^2 + y^2 = a^2$ where a, b>0 are shown on the diagram below:



MP is perpendicular to *Ox* and *P* is a point on the hyperbola in the first quadrant.

(ii) Show that the co-ordinates of *P* are $(a\sec\theta, b\tan\theta)$. **2**

(iii) If *Q* is another point on the hyperbola with co-ordinates $(a\sec\phi, b\tan\phi)$ where $\theta + \phi = \dfrac{\pi}{2}$ and $\theta \neq \dfrac{\pi}{4}$, show that the equation of the chord *PQ* is $y = \dfrac{b}{a}(\cos\theta + \sin\theta)x - b$. **3**

(iv) Show that every such chord passes through a fixed point and determine its co-ordinates. **1**

(v) State the equation of the asymptotes for the hyperbola $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$. **1**

(vi) Show that as $\theta \to \dfrac{\pi}{2}$, the chord *PQ* approaches a line parallel to an asymptote. **1**

End of Paper

STANDARD INTEGRALS

$$\int x^n\,dx = \frac{1}{n+1}x^{n+1}, \quad n \neq -1;\ x \neq 0,\ \text{if } n < 0$$

$$\int \frac{1}{x}\,dx = \ln x,\ x > 0$$

$$\int e^{ax}\,dx = \frac{1}{a}e^{ax},\ a \neq 0$$

$$\int \cos ax\,dx = \frac{1}{a}\sin ax,\ a \neq 0$$

$$\int \sin ax\,dx = -\frac{1}{a}\cos ax,\ a \neq 0$$

$$\int \sec^2 ax\,dx = \frac{1}{a}\tan ax,\ a \neq 0$$

$$\int \sec ax \tan ax\,dx = \frac{1}{a}\sec ax,\ a \neq 0$$

$$\int \frac{1}{a^2+x^2}\,dx = \frac{1}{a}\tan^{-1}\frac{x}{a},\ a \neq 0$$

$$\int \frac{1}{\sqrt{a^2-x^2}}\,dx = \sin^{-1}\frac{x}{a},\ a > 0,\ -a < x < a$$

$$\int \frac{1}{\sqrt{x^2-a^2}}\,dx = \ln\left(x+\sqrt{x^2-a^2}\right),\ x > a > 0$$

$$\int \frac{1}{\sqrt{x^2+a^2}}\,dx = \ln\left(x+\sqrt{x^2+a^2}\right)$$

NOTE : $\ln x = \log_e x, \qquad x > 0$

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Extension 2 Mathematics
Multiple Choice Answer Sheet

Student Number__________

Completely fill the response oval representing the most correct answer.

1.	**A** ⬭	**B** ⬭	**C** ⬭	**D** ⬭
2.	**A** ⬭	**B** ⬭	**C** ⬭	**D** ⬭
3.	**A** ⬭	**B** ⬭	**C** ⬭	**D** ⬭
4.	**A** ⬭	**B** ⬭	**C** ⬭	**D** ⬭
5.	**A** ⬭	**B** ⬭	**C** ⬭	**D** ⬭
6.	**A** ⬭	**B** ⬭	**C** ⬭	**D** ⬭
7.	**A** ⬭	**B** ⬭	**C** ⬭	**D** ⬭
8.	**A** ⬭	**B** ⬭	**C** ⬭	**D** ⬭
9.	**A** ⬭	**B** ⬭	**C** ⬭	**D** ⬭
10.	**A** ⬭	**B** ⬭	**C** ⬭	**D** ⬭

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