Mrs Collett
Mrs Kerr

Name: ....

Teacher:..

Pymble Ladies' College

HIGHER SCHOOL CERTIFICATE

TRIAL EXAMINATION

2014

Mathematics Extension 2

Time Allowed: 3 hours

General Instructions

- Reading time – 5 minutes.
- Working time – 3 hours.
- Write using black or blue pen. Black pen is preferred.
- Board approved calculators may be used.
- A table of standard integrals is provided at the back of this paper.
- In Questions 11-16, show relevant mathematical reasoning and/or calculations.
- Start each question in a **new** booklet.

Total Marks – 100

Section I Pages 1-4

10 marks

- Attempt all Questions 1-10
- Allow about 15 mins for this section

Section II Pages 5-12

90 marks

- Attempt Questions 11-16
- Allow about 2 hour 45 minutes for this section

Mark	**/100**
Highest Mark	**/100**
Rank	

Section I

10 marks
Attempt Questions 1-10
Allow about 15 minutes for this section.

Use the multiple choice answer sheet for Questions 1-10.

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1 If $(a+bi)^2 = i$, then what are possible values for $a, b \in \mathbb{R}$?

(A) $a = \frac{1}{4}, \ b = \frac{1}{4}$

(B) $a = -\frac{1}{\sqrt{2}}, \ b = \frac{1}{\sqrt{2}}$

(C) $a = \frac{1}{\sqrt{2}}, \ b = \frac{1}{\sqrt{2}}$

(D) $a = \frac{1}{2}, \ b = \frac{1}{2}$

2 The polynomial $P(x) = x^3 + 3x^2 - 24x + 28$ has a double zero.

What is the value of the double zero?

(A) -7

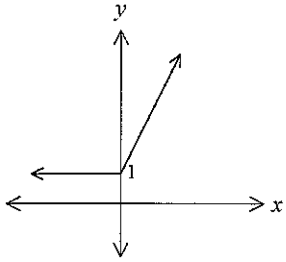
(B) -4

(C) 4

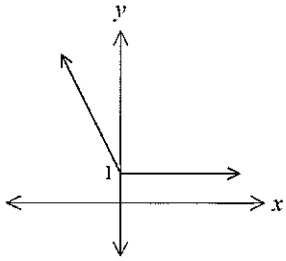
(D) 2

3 Which graph shows $y = 1 + x + |x|$?

(A)



(B)



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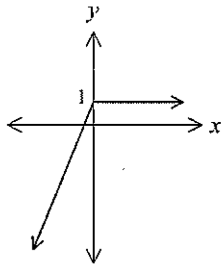
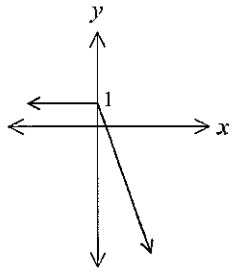


(D)



4 The graph of the ellipse $\dfrac{(x-1)^2}{9} + \dfrac{y^2}{4} = 1$ and the graph of the hyperbola $x^2 - y^2 = 4$ have

(A) no points in common.

(B) 1 point in common.

(C) 2 points in common.

(D) 3 points in common.

5 $\displaystyle\int \sin^{-1} 2x \, dx =$

(A) $x\sin^{-1}2x + \frac{1}{4}\sqrt{1-4x^2} + C, \quad |x| \ge -1$

(B) $x\sin^{-1}2x - \frac{1}{4}\sqrt{1-4x^2} + C, \quad |x| \ge -1$

(C) $x\sin^{-1}2x + \frac{1}{2}\sqrt{1-4x^2} + C, \quad |x| \ge -1$

(D) $x\sin^{-1}2x - \frac{1}{2}\sqrt{1-4x^2} + C, \quad |x| \ge -1$

6 Which of the following would be neither odd nor even?

(A) $y = x^2 \sin x$

(B) $y = \sin\left(x^2\right)$

(C) $y = \left(\sin x\right)^2$

(D) $y = x^2 + \sin x$

7 What is the exact value of $\left(\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}i\right)^{100}$?

(A) 1

(B) -1

(C) $\frac{1}{2^{50}}$

(D) $-\frac{1}{2^{50}}$

8 If $\frac{3x-19}{(x+3)(2x-1)} = \frac{a}{x+3} + \frac{b}{2x-1}$, then find the values of a and b.

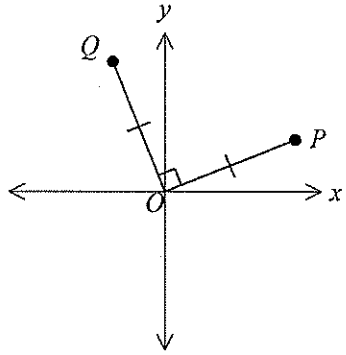
(A) $a = -4, \quad b = 5$

(B) $a = -4, \quad b = -5$

(C) $a = 4, \quad b = 5$

(D) $a = 4, \quad b = -5$

9 On the diagram P and Q represent complex numbers z and w respectively. Triangle OPQ is right angled and isosceles.



Which of the following is **false**?

(A) $|z|^2 + |w|^2 = |z+w|^2$

(B) $z^2 - w^2 = 0$

(C) $z^2 + w^2 = 0$

(D) $w = iz$

10 The ellipse $x^2 + 2ax + 2y^2 + 4by + 16 = 0$ has its centre at $(3, -2)$. Find the values of a and b.

(A) $a = -3,\ b = -2$

(B) $a = 2,\ b = -3$

(C) $a = -3,\ b = 2$

(D) $a = 3,\ b = 2$

Section II

90 marks
Attempt Questions 11-16
Allow about 2 hours and 45 minutes for this section.

Answer each question in the appropriate writing booklet. Extra booklets are available.

In Questions 11 – 16, your responses should include relevant mathematical reasoning and/or calculations.

Question 11. (15 marks). Use a **Separate** Booklet. **Marks**

(a) Use the substitution $u = 4 + \sin x$ to find **2**

$$\int \frac{\sin x \cos x}{4 + \sin x}\, dx.$$

(b) Let $w = -1 + \sqrt{3}\, i$ and $z = 1 - i$.

(i) Find wz in the form $a + ib$. **1**

(ii) Find w and z in mod-arg form. **2**

(iii) Hence, find the exact value of $\sin\frac{5\pi}{12}$. **2**

(c) Let polynomial $P(x) = ax^6 - bx^5 + 1$.

(i) State the conditions for α to be a zero of multiplicity two of $P(x)$. **1**

(ii) Given that $P(x)$ is divisible by $(x+1)^2$ find a and b. **3**

(d) The line $x = 1$ is a directrix and the point $(2,0)$ is a focus of the conic whose eccentricity is $\sqrt{2}$.

(i) Derive the equation of the conic. **3**

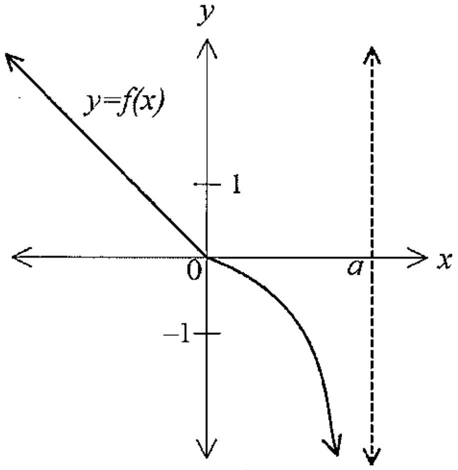
(ii) Prove that it is a rectangular hyperbola. **1**

End of Question 11

Question 12. (15 marks) Use a **Separate** Booklet. **Marks**

(a) Find $\int \frac{\ln x}{x^2}\,dx$. **2**

(b) The graph of the function $y = f(x),\ x < a$ is shown below.



Sketch the following curves on separate half-page diagrams.

(i) $y = |f(x)|$. **1**

(ii) $y = f(|x|)$. **1**

(iii) $y = \frac{1}{f(x)}$. **2**

(c) Let C be the curve $3e^{x-y} = x^2 + y^2 + 1$. **3**
Find the equation of the tangent to C at the point $(1,1)$.

Question 12 continues on page 7

(d) (i) Expand $(a-b)^3$. **1**

(ii) Solve $z^3=-1$. **2**

(iii) Express the polynomial $z^3-3iz^2-3z+1+i$ in the form $(z+p)^3+q$ where p is an imaginary number and q is a real number. **1**

(iv) Hence solve $z^3-3iz^2-3z+1+i=0$ giving the solution in the form $z=x+iy$ where $x,y\in\mathbb{R}$. **2**

End of Question 12

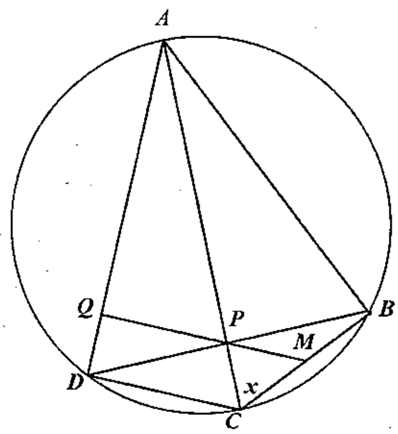
Question 13. (15 marks) Use a **Separate** Booklet. **Marks**

(a) When the polynomial $p(x) = x^4 + ax + 2$ is divided by $x^2 + 1$ the remainder is $2x + 3$. Find the value of a. **2**

(b) Using the substitution $t = \tan\frac{x}{2}$, find $\int \frac{\tan x}{1 + \cos x}\,dx$. **3**

(c) Consider the region bounded by the curve $y = x^2 - 6x + 8$ and the x-axis. Use the method of cylindrical shells to find the volume of the solid formed if the region is rotated about the y-axis to form a solid of revolution. **3**

(d) $ABCD$ is a cyclic quadrilateral. Diagonals AC and BD intersect at right angles at P. M is the midpoint of BC. MP produced meets AD at Q. Let $\angle MCP = x$.

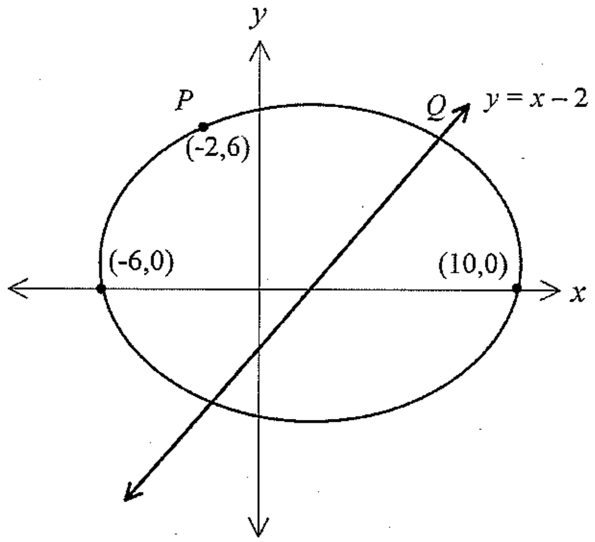


(i) Show $\angle MCP = \angle CPM$. **2**

(ii) Show $MQ \perp AD$. **2**

Question 13 continues on page 9

(e) The ellipse shown below passes through point $P(-2,6)$. **3**
The centre of the ellipse lies on the x-axis , and the ellipse passes through the points $(-6,0)$ and $(10,0)$.



The line shown is $y = x - 2$. This line intersects the ellipse at Q.

What is the x coordinate of point Q?

End of Question 13

Question 14. (15 marks) Use a **Separate** Booklet. **Marks**

(a) (i) Derive the equation of the tangent to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ at point $P(x_1, y_1)$. **3**

(ii) The tangents to the ellipse $x^2+4y^2=4$ at the points $P(2\cos\theta, \sin\theta)$ and $Q(2\cos\phi, \sin\phi)$ are at right angles to each other. **2**

Show that $4\tan\theta\tan\phi=-1$.

(b) If w is one of the complex roots of $z^3=1$, simplify **3**

$(1-w)(1-w^2)(1-w^4)(1-w^8)$.

(c) (i) If $I_n=\int_0^{\frac{\pi}{4}}\tan^n x\,dx$ prove that $I_n+I_{n-2}=\frac{1}{n-1}$, $n>2$. **2**

(ii) Hence, evaluate $\int_0^{\frac{\pi}{4}}\tan^5 x\,dx$. **2**

(d) Sketch the locus of z if $\frac{z-2i}{z-1}$ is purely imaginary. **3**

End of Question 14

Question 15. (15 marks) Use a **Separate** Booklet. **Marks**

(a) The base of a solid is given by the region in the xy-plane enclosed by the curve $y = x^2$ and $y = 8 - x^2$.
Each cross-section perpendicular to the x-axis is a square.

(i) Show that the area of the square cross-section at $x = h$ is $\left(8 - 2h^2\right)^2$. **1**

(ii) Hence, find the volume of the solid. **3**

(b) Show that $\displaystyle\int_0^1 \frac{dx}{x^2 - x + 1} = \frac{2\sqrt{3}\pi}{9}$. **2**

(c) Let $f(x) = \dfrac{4}{x-1} - \dfrac{4}{x+1} - 1$, where $x \neq \pm 1$.

(i) Find the x and y intercepts of the graph of $y = f(x)$. **2**

(ii) Show that $y = f(x)$ is an even function. **1**

(iii) Find the equation of the horizontal asymptote. **1**

(iv) Sketch the graph of $y = f(x)$. **2**

(v) Let S be the area bound by the graph of $y = f(x)$, the straight lines $x = 3$, $x = a$ $(a > 3)$ and $y = -1$. **3**

Find S in terms of a and deduce that $S < 4\ln 2$.

End of Question 15

Question 16. (15 marks) Use a **Separate** Booklet. **Marks**

(a) The locus of w is described by the equation $|w+3| = |w-2+5i|$.

(i) Sketch on an Argand Diagram the locus of w. **2**

(ii) Find the Cartesian equation of the locus of w. **2**

(b) (i) Given that $\frac{1}{n} - \frac{1}{n+1} = \frac{1}{n(n+1)}$ explain why $\frac{1}{(n+1)^2} < \frac{1}{n(n+1)}, n \in \mathbb{Z}^+$. **1**

(ii) Using induction, prove $S_n = \sum_{r=1}^{n} \frac{1}{r^2} \leq 2 - \frac{1}{n}, n \geq 1$. **3**

(c) Consider the quadratic equation $x^2 - x + k = 0$ where k is a real number.
The equation has 2 distinct positive roots α and β.

(i) Show $0 < k < \frac{1}{4}$. **2**

(ii) Show that $\frac{1}{\alpha^2} + \frac{1}{\beta^2} > 8$. **2**

(d) Given $I = \int_{-1}^{1} \frac{x^2 e^x}{e^x + 1} dx$ and $J = \int_{-1}^{1} \frac{x^2}{e^x + 1} dx$.

(i) Use the substitution $u = -x$ in I to show $I = J$. **1**

(ii) Hence evaluate I and J. **2**

End of Paper