

Pymble ladies 2005 Ext. 2 trial

Question 1 (15 marks) Use a separate writing booklet

MARKS

(a) Find $\int \tan 2x \sec 2x \, dx$. 1

(b) Find $\int \frac{1}{x} \sec^2(\ln x) \, dx$. 1

(c) Find $\int \frac{4x - x^2}{(x+1)(x^2+4)} \, dx$. 3

(d) Find $\int \cos 5x \sin 2x \, dx$. 2

(e) Evaluate $\int_0^{\frac{\pi}{3}} \frac{1}{1 - \sin x} \, dx$ using the substitution $t = \tan x$. 3

(f) Find $\int_0^{\frac{\pi}{2}} x \sin x \cos x \, dx$. 3

(g) Using the result $\int_0^a f(x) \, dx = \int_0^a f(a-x) \, dx$, show that

$$\int_0^{\frac{\pi}{2}} \frac{\cos x}{\cos x + \sin x} \, dx = \frac{\pi}{4}. \quad 2$$

Question 2 (14 marks) Use a separate writing booklet

MARKS

- (a) Use the graph of $y = \ln x$ to sketch the graphs of:
- (i) $y = \ln(-x)$ 1
 - (ii) $y = -\ln x$ 1
 - (iii) $y = |\ln x|$ 1
- (b) Use the graphs of $y = x$ and $y = e^{-x}$ to sketch the graph of $y = xe^{-x}$. 2
- (c) Use the graph of $y = x^2 - 1$ to sketch the graph of $y = (x^2 - 1)^2$. 2
- (d) For the function $f(x) = 3x - \frac{x^3}{4}$, use the graph of $y = f(x)$ to sketch the graph of $y^2 = f(x)$. 2
- (e) Use the graphs of $y = 2^u$ and $u = \cos x$ ($0 \leq x \leq 2\pi$) to sketch the graph of $y = 2^{\cos x}$ ($0 \leq x \leq 2\pi$). 2
- (f) Sketch the graph of $y = \sin 2x$ for $0 \leq x \leq 2\pi$. Use this graph to solve the inequality $|\sin 2x| \geq \frac{1}{2}$, for $0 \leq x \leq 2\pi$. 3

Question 3 (15 marks) Use a separate writing booklet

MARKS

- (a) Solve for z where $z \in \mathbb{C}$

2

$$z^2 + 2iz + 2 = 0.$$

- (b) Form a quadratic equation whose roots are $4i$ and $3+i$.

2

- (c) If $w = 1 + 2i$ and $z = 2 - 3i$, express in the form $a + ib$.

(i) $w + z$

1

(ii) $w\bar{z}$

2

(iii) $\frac{w}{z}$

2

- (d) Express $\sqrt{3} - i$ in the form $r(\cos\theta + i\sin\theta)$ and plot on the Argand diagram showing θ , r and the Cartesian coordinates.

3

- (e) By expanding $(\cos\theta + i\sin\theta)^4$, find expressions for $\cos 4\theta$ and $\sin 4\theta$ in terms of powers of $\cos\theta$ and $\sin\theta$. Hence deduce an expression for $\tan 4\theta$ in terms of powers of $\tan\theta$.

3

Question 4 (15 marks) Use a separate writing booklet**MARKS**

- (a) For the ellipse $\frac{x^2}{16} + \frac{y^2}{25} = 1$ find:
- (i) the eccentricity; 1
 - (ii) the coordinates of the foci; 1
 - (iii) the equations of the directrices. 1
 - (iv) Sketch the ellipse showing essential features. 1
- (b) Find the equation of the tangent to the hyperbola $\frac{x^2}{12} - \frac{y^2}{27} = 1$ at the point (4, 3). 2
- (c) A point $P(a \sec \theta, b \tan \theta)$ lies on the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$. 4
- The line through P perpendicular to the x -axis meets an asymptote at Q and the normal at P meets the x -axis at N .
Show that QN is perpendicular to the asymptote.
- (d) The point $P\left(ct, \frac{c}{t}\right)$ lies on the rectangular hyperbola $xy = c^2$. 5
- The normal at P meets the hyperbola again at Q .
 M is the midpoint of PQ . Find the equation of the locus of M .

Question 5 (15 marks) Use a separate writing booklet

MARKS

- (a) Find $P(x)$, given that $P(x)$ is monic, of degree 3, with 5 as a single zero and -2 as a zero of multiplicity 2. 2
- (b) Find the remainder when $P(x) = x^3 + 2x^2 + 1$ is divided by $x + i$. 1
- (c) If $P(x) = x^4 - 2x^3 - x^2 + 6x - 6$ has a zero $1 - i$, find the zeros of $P(x)$ over \mathbb{C} , and factorise $P(x)$ fully over \mathbb{R} . 3
- (d) Solve the equation $18x^3 + 27x^2 + x - 4 = 0$, given the roots are in arithmetic progression. 3
- (e) The equation $x^3 + x^2 - 2x - 3 = 0$ has roots α , β and γ . Find the equations with roots:
- (i) $\frac{\alpha}{2}, \frac{\beta}{2}, \frac{\gamma}{2}$; 1
- (ii) $\alpha + 2, \beta + 2$ and $\gamma + 2$. 1
- (f) The equation $x^3 + x^2 + 2 = 0$ has roots α , β and γ . Evaluate:
- (i) $\alpha^3 + \beta^3 + \gamma^3$ 2
- (ii) $\alpha^4 + \beta^4 + \gamma^4$ 2

Question 6 (15 marks) Use a separate writing booklet

MARKS

- (a) If $I_n = \int_0^{\frac{\pi}{2}} x^n \cos x \, dx$ for $n \geq 0$, show that $I_n = \left(\frac{\pi}{2}\right)^n - n(n-1)I_{n-2}$ for $n \geq 2$. Hence evaluate I_6 . **5**
- (b) By taking slices perpendicular to the axis of rotation, use the method of slicing to find the volume of the solid obtained by rotating the region, determined by $0 \leq x \leq 2$ and $0 \leq y \leq x^3$, about the line $y = 8$. **5**
- (c) (i) Let R be the region in the plane for which $0 \leq x \leq \frac{\pi}{2}$ and $0 \leq y \leq \sin x$. Sketch R . **1**
- (ii) A solid is formed by rotating the region R about the y -axis. Use the method of cylindrical shells to find the volume of the solid. **4**

Question 7 (15 marks) Use a separate writing booklet

MARKS

- (a) The rise and fall of the tide at Bedrock Harbour may be taken as simple harmonic, the interval between successive high tides being $12\frac{1}{2}$ hours. The harbour entrance has a depth of 15m at high tide and 7m at low tide. 7

If low tide occurs at 11am on a certain day, find the earliest time thereafter that a ship requiring a minimum depth of 13m of water can pass through the entrance.

- (b) Use Mathematical Induction to prove DeMoivre's Theorem ie. 4
 $(\cos \theta + i \sin \theta)^n = \cos n\theta + i \sin n\theta$ for all positive integer values of n .

(c) Let $f(x) = \begin{cases} \frac{\sin x}{x} & \text{for } 0 < x < \frac{\pi}{2} \\ 1 & \text{for } x = 0 \end{cases}$

- (i) Find the derivative of $f(x)$ for $0 < x < \frac{\pi}{2}$ and prove that 2
 $f'(x)$ is negative in this interval.

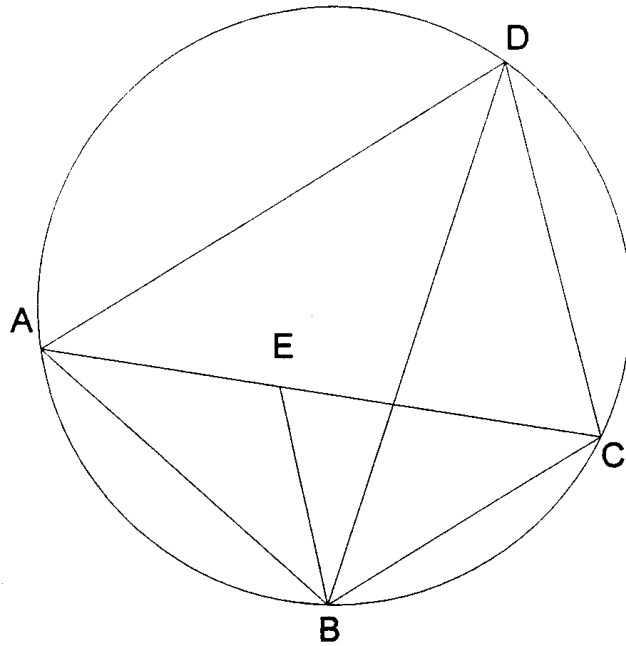
- (ii) Sketch the graph of $y = f(x)$ for $0 < x < \frac{\pi}{2}$ and deduce that 2
 $\sin x > \frac{2x}{\pi}$ in this interval.

Question 8 (15 marks)

Use a separate writing booklet

MARKS

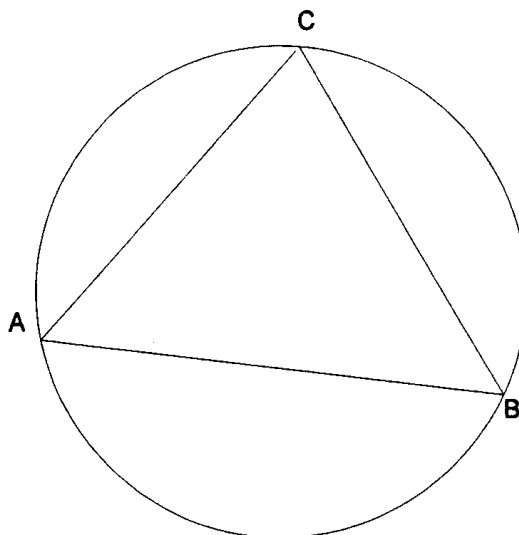
(a)



In the diagram $ABCD$ is a cyclic quadrilateral. E is a point on AC such that $\angle ABE = \angle DBC$.

- (i) Show that $\triangle ABE \sim \triangle DBC$ and $\triangle ABD \sim \triangle EBC$. 2
- (ii) Hence show that $AB \cdot DC + AD \cdot BC = AC \cdot DB$ 2

(iii)



In the diagram ABC is an equilateral triangle inscribed in a circle. P is a point on the minor arc AB of the circle. Use the result in part (ii) to show that $PC = PA + PB$.

2

Question 8 (continued)

MARKS

- (b) (i) Prove that $a^2 + b^2 \geq 2ab$ where a, b are any two real numbers. 2
- (ii) If a, b and c are three real, positive numbers all less than 1, such that $a + b + c > abc$, prove that $a^2 + b^2 + c^2 > abc$. 2

- (c) When a particle is projected vertically upwards from the moon's surface, its distance x from the centre of the moon is given by

$$\frac{d}{dx} \left(\frac{1}{2} v^2 \right) = -f \frac{R^2}{x^2}$$

where v is the upward speed, R is the radius of the moon and f is the acceleration due to gravity at the moon's surface and any possible atmospheric resistance is neglected. If v_0 is the speed of projection, show that:

- (i) $v^2 = \frac{2f R^2}{x} + v_0^2 - 2f R$; 2
- (ii) the maximum height H , above the moon's surface, to which the particle will ascend is given by 2
- $$H = \frac{R v_0^2}{2f R - v_0^2}.$$
- (iii) Taking $R \approx 1800 \text{ km}$, $f \approx 1.6 \text{ ms}^{-2}$, estimate the escape velocity of the particle from the moon in kms^{-1} . 1