Rollan 2004 Ext. 2 trial

Question One (15 marks) (Use a SEPARATE writing booklet) **Marks**

(a) Find $\int \frac{1}{\sqrt{7-6x-x^2}}dx$ 2

(b) (i) Find real numbers a, b such that

$$\frac{x}{(x-1)(x+4)} = \frac{a}{x-1} + \frac{b}{x+4}$$

(ii) Hence find $\int \frac{x}{(x-1)(x+4)}dx$ 4

(c) Use the substitution $u = x-2$ to find the exact value of $\int_1^3 x(x-2)^5\,dx$ 4

(d) Evaluate $\int_0^{\frac{\pi}{6}} \sec 4x \tan 4x\,dx$ 2

(e) Evaluate $\int_0^{\frac{\pi}{6}} x\cos x\,dx$ 3

Question Two (15 marks) (Use a SEPARATE writing booklet) **Marks**

(a) Given the complex number $z = 1 - i\sqrt{3}$, find

(i) $|z| + \dfrac{1}{|z|}$ **2**

(ii) $\arg \bar{z}$ **2**

(b) Write $\left(1+\sqrt{3}i\right)^{-1}$ in modulus/argument form. **2**

(c) Sketch the graph of $\arg\left(\dfrac{z-2}{z+2i}\right) = \dfrac{\pi}{2}$ **3**

(d) When $z_1 = 2$, $z_2 = i$, show geometrically how to construct the vectors representing

(i) $z_1 + z_2$ **1**

(ii) $z_1 - z_2$ **1**

(e) $z = x + iy$ is such that $\dfrac{z-i}{z+1}$ is purely imaginary. Find the equation of the locus of the point P representing z and show this locus on an Argand diagram. **4**

Question Three (15 marks) (Use a SEPARATE writing booklet) **Marks**

(a) (i) Sketch the graphs of $y = \log_2 x$ and $y = \dfrac{1}{x}$. **2**

(ii) Hence sketch the graph of $y = \log_2 x + \dfrac{1}{x}$ **1**

(b) Sketch the graph of $y = f(x)$, where $f(x) = \dfrac{x^2}{x^2-1}$, showing all the main features. Hence sketch the graphs of: **2**

(i) $y = \dfrac{1}{f(x)}$ **2**

(ii) $y = |f(x)|$ **2**

(iii) $y = \sqrt{f(x)}$ **2**

(iv) $y = \log_2\left[f(x)\right]$ **2**

(v) $y = f^{-1}(x)$, by first restricting the domain of $y = f(x)$. **2**

Question Four (15 marks) (Use a SEPARATE writing booklet) **Marks**

(a) For the hyperbola $\frac{y^2}{9}-\frac{x^2}{16}=1$ find

(i) the eccentricity, 1

(ii) the coordinates of the foci, 1

(iii) the equations of the directrices, 1

(iv) the equations of the asymptotes. 1

Sketch the hyperbola. 1

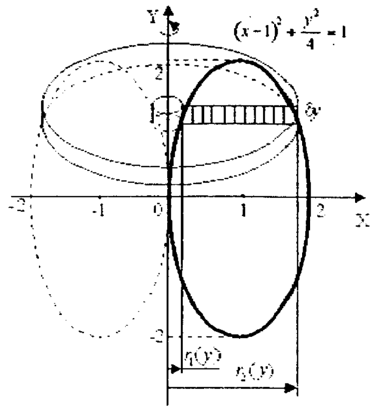
(b) The points $P(a\cos\theta,\ b\sin\theta)$ and $Q(a\cos\phi,\ b\sin\phi)$ lie on the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ and the chord PQ subtends a right angle at $(0,0)$. Show that $\tan\theta\tan\phi=-\frac{a^2}{b^2}$. 3

(c) The point $P(a\cos\theta,\ b\sin\theta)$ lies on the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$. The tangent at P cuts the y-axis at B and Y is the foot of the perpendicular from P to the y-axis. Show that $OY\cdot OB=b^2$. 3

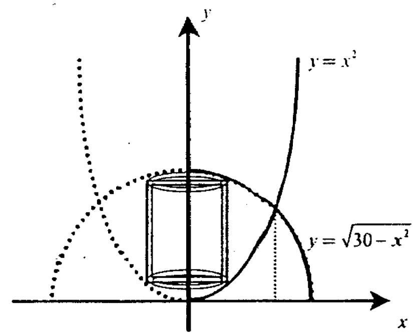
(d) The points $P\left(cp,\frac{c}{p}\right)$ and $Q\left(cq,\frac{c}{q}\right)$ lie on the rectangular hyperbola $xy=c^2$. The chord PQ subtends a right angle at the another point $R\left(cr,\frac{c}{r}\right)$ on the hyperbola. Show that the normal at R is parallel to PQ. 4

Question Five (15 marks) (Use a SEPARATE writing booklet) **Marks**

(a) By taking slices perpendicular to the axis of rotation, use the method of slicing to find the volume of the solid obtained by rotating the region enclosed within the ellipse $(x-1)^2+\frac{y^2}{4}=1$ about the y-axis. 6



(b) The shaded area enclosed by the curves $y=x^2$ and $y=\sqrt{30-x^2}$ as well as the y-axis as shown in the diagram is rotated around the y-axis.
By first calculating the x co-ordinate of their point of intersection, use the method of cylindrical shells to calculate the volume of the solid formed. 6



(c) If $\frac{a}{c}=\frac{a-b}{b-c}$, then b is the Harmonic Mean of a and c.

(i) Prove that $b=\frac{2ac}{a+c}$ 1

(ii) Prove that the reciprocals of a, b, c form an Arithmetic Progression. 2

Question Six (15 marks) (Use a SEPARATE writing booklet) **Marks**

(a) Express $P(x) = x^4 - 3x^3 - 2x^2 - 6x - 8$ as a product of irreducible factors over

(i) The set of rational numbers, **3**

(ii) The set of complex numbers. **1**

(b) (i) If $P(x)$ is divided by (x^2+1), explain why the most common form of the remainder $R(x)$ is $ax+b$. **1**

(ii) If $P(i) = -1 - i$ and $P(-i) = -1 + i$, calculate the values of a and b. **3**

(c) If $a>0, b>0$ and $c>0$, show that $(a+b+c)\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right) \geq 9$. **4**

(d) If $y = uv$, then

$$\frac{dy}{dx} = v\frac{du}{dx} + u\frac{dv}{dx} \text{ and}$$

$$\frac{d^2y}{dx^2} = v\frac{d^2u}{dx^2} + \frac{du}{dx}\frac{dv}{dx} + \frac{du}{dx}\frac{dv}{dx} + u\frac{d^2v}{dx^2}$$

$$= \frac{d^2u}{dx^2}v + 2\frac{du}{dx}\frac{dv}{dx} + u\frac{d^2v}{dx^2}$$

(i) Derive a similar expression for $\frac{d^3y}{dx^3}$ **2**

(ii) Hence postulate an expression for $\frac{d^ny}{dx^n}$ **1**

Question Seven (15 marks) (Use a SEPARATE writing booklet) **Marks**

(a) A sequence $t_1, t_2, t_3, \ldots, t_n, \ldots$ is defined by

$t_1 = 1$,

$3t_{n+1} = 2t_n - 1$, for $n \geq 2$.

(i) Prove by mathematical induction that $t_n = 3\left(\frac{2}{3}\right)^n - 1$ for all integers $n \geq 1$. **3**

(ii) Hence find $\sum_{r=1}^{n} t_r$. **2**

(b) A particle P of mass m kg moves vertically in a medium in which the resistance to motion has magnitude $\frac{mv}{10}$ when the speed of the particle is v ms^{-1}. The acceleration due to gravity is 10 ms^{-2}.

(i) If the particle falls vertically from rest, show that its acceleration a ms^{-2} is given by $a = 10 - \frac{v}{10}$. Hence show that the terminal velocity of the particle is 100 ms^{-1}. **2**

(ii) The particle is projected vertically upwards with speed 100 ms^{-1}.

(1) Show that its acceleration a ms^{-2} is given by $a = -10 - \frac{v}{10}$. **1**

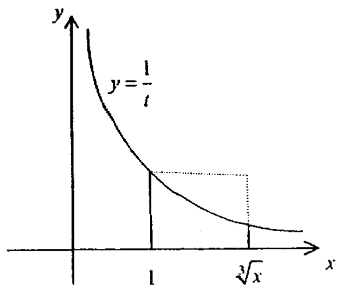
(2) Find expressions for the speed v ms^{-1} and the height x metres of the particle after time t seconds. **5**

(3) Find the time taken by the particle to reach its maximum height, and the maximum height attained. **2**

Question Eight (15 marks) (Use a SEPARATE writing booklet) **Marks**

(a) The diagram represents the curve

$y = \frac{1}{t}$ for $t > 0$.



(i) If $x > 1$, show that $\int_1^{\sqrt[3]{x}} \frac{1}{t}\,dt = \frac{1}{3}\ln x$ **2**

(ii) Show that for $x > 1$, $0 < \frac{1}{3}\ln x < \sqrt[3]{x}$ **2**

(b) (i) If $I_n = \int x^n e^{ax}\,dx$, show that $I_n = \frac{x^n e^{ax}}{a} - \frac{n}{a} I_{n-1}$ **3**

(ii) Hence evaluate I_2 **3**

(c) The points P, Q, R which represent the complex numbers z_1, z_2, z_3 lie on a circle through the origin on the Argand diagram.

Show that the points which represent $\frac{1}{z_1}, \frac{1}{z_2}, \frac{1}{z_3}$ are collinear. **5**