SYDNEY BOYS HIGH SCHOOL
MOORE PARK, SURRY HILLS

2014
HIGHER SCHOOL CERTIFICATE
TRIAL PAPER

Mathematics Extension 2

General Instructions

- Reading Time – 5 Minutes
- Working time – 3 Hours
- Write using black or blue pen. Pencil may be used for diagrams.
- Board approved calculators maybe used.
- Answer Questions 1 to 10 on the sheet provided.
- Each Question from 11 to 16 is to be returned in a separate bundle.
- All necessary working should be shown in every question

Total Marks – 100

- Attempt questions 1 – 16
- Answer in simplest exact form unless otherwise instructed

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- NOTE: This is a trial paper only and does not necessarily reflect the content or format of the final Higher School Certificate examination paper for this subject.

Use Multiple Choice Answer Sheet

Question 1

Seven people are to be placed in four hotel rooms.
In how many ways may this be done?

A: 4^7

B: ${}^7C_4$

C: ${}^7P_4$

D: 7^4

Question 2

$i^{2114} =$

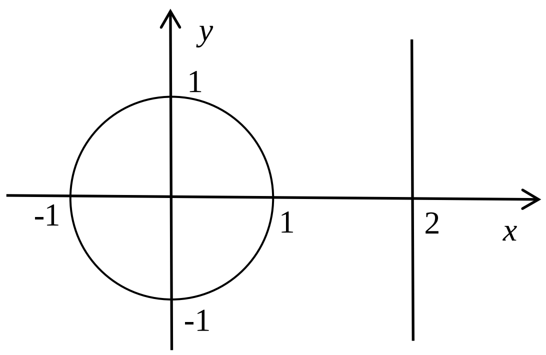
A: 1

B: i

C: $-i$

D: -1

Question 3



The circle $x^2 + y^2 = 1$ is rotated about the line $x = 2$. With use of cylindrical shells, the volume is given by:

A: $4\pi\int_{-1}^{1}(2-x)\sqrt{1-x^2}\,dx$

B: $8\pi\int_{0}^{1}(2-x)\sqrt{1-x^2}\,dx$

C: $2\pi\int_{-1}^{1}(2-x)\sqrt{1-x^2}\,dx$

D: $4\pi\int_{1}^{2}(2-x)\sqrt{1-x^2}\,dx$

Question 4

The equation of the chord of contact from $(5,-2)$ to the ellipse $\frac{x^2}{16}+\frac{y^2}{9}=1$ is given by:

A: $\frac{x}{8}-\frac{5y}{16}=1$

B: $\frac{5x}{16}+\frac{2y}{9}=1$

C: $\frac{5x}{16}-\frac{2y}{9}=0$

D: $\frac{5x}{16}-\frac{2y}{9}=1$

Question 5

The roots of $x^3+5x+11=0$ are $\alpha, \beta,$ and γ.

The value of $\alpha^2\beta\gamma+\alpha\beta^2\gamma+\alpha\beta\gamma^2$ is:

A: 25

B: 0

C: -55

D: 55

Question 6

If a and b are positive, which of the following is false?

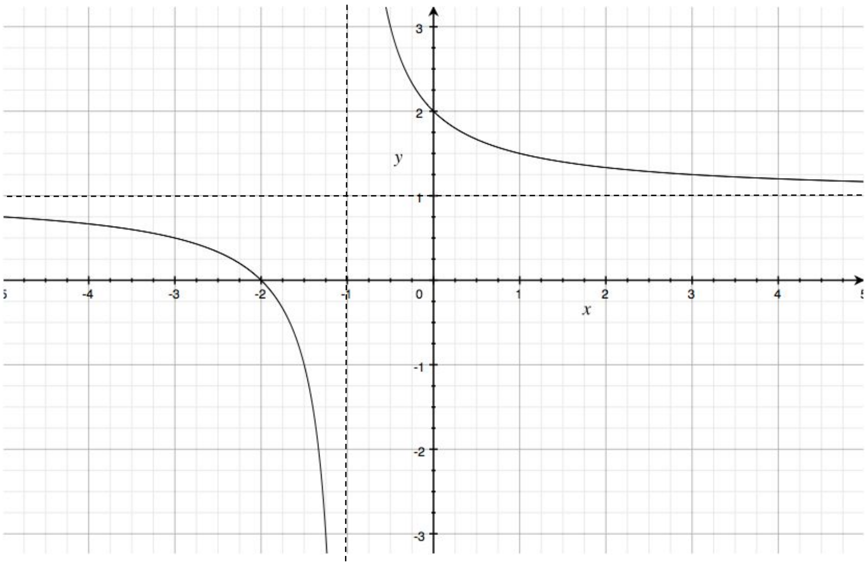
A: $\frac{a}{b}+\frac{b}{a}\geq 2$.

B: $\frac{a+b}{2}\leq\sqrt{ab}$.

C: $\left(\sqrt{a}-\sqrt{b}\right)^2\geq 2ab$.

D: $(a+b)^2\geq(a-b)^2+(2ab)^2$.

Question 7



The graph has equation:

A: $(x-1)(y+1)=1$

B: $y=\dfrac{x+2}{x}$

C: $(x+1)(y-1)=1$

D: $y=\dfrac{x}{x+1}$

Question 8

$1+i$ is a zero of x^3+ax+b where *a*, *b* are real, therefore the values of *a* and *b* are:

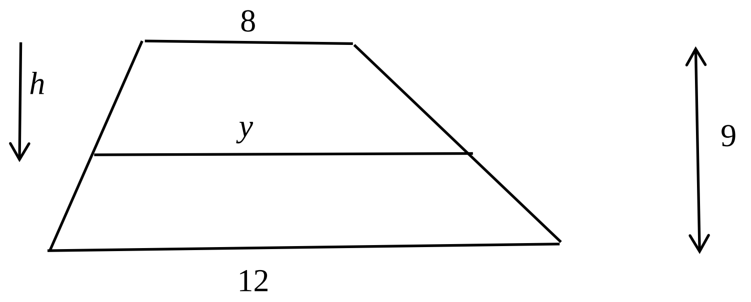
A: $a=-2,\ \ b=-4$

B: $a=-2,\ \ b=4$

C: $a=2,\ \ b=-4$

D: $a=2,\ \ b=4$

Question 9



The diagram shows a trapezium, with an internal parallel line. Which of the following is true?

A: $y=\frac{3}{4}h+8$.

B: $y=\frac{3}{4}h+9$.

C: $4y=9h+72$

D: $9y=4h+72$

Question 10

By considering the graphs of $y=3x^2-2x-2$ and $y=|3x|$, the solution to $3x^2-2x-2\le|3x|$ is:

A: $-\frac{1}{3}\le x\le 2$.

B: $-1\le x\le \frac{3}{2}$.

C: $-\frac{1}{3}\le x\le \frac{3}{2}$

D: $-1\le x\le 2$

Question 11. (15 marks) **(Start a new answer booklet.)**

Marks

(a) Given $z = 1 - i$, find the values of w such that **2**

$$w^2 = i + 3\bar{z}$$

(b) On separate Argand diagrams, shade the following regions:

(i) $4 \le z + \bar{z} \le 10$ **1**

(ii) $\arg\left(z^2\right) = \frac{2\pi}{3}$ **1**

(iii) $z\bar{z} = 4$ **1**

(c) (i) Show that $z = 1 + i$ is a root of the polynomial

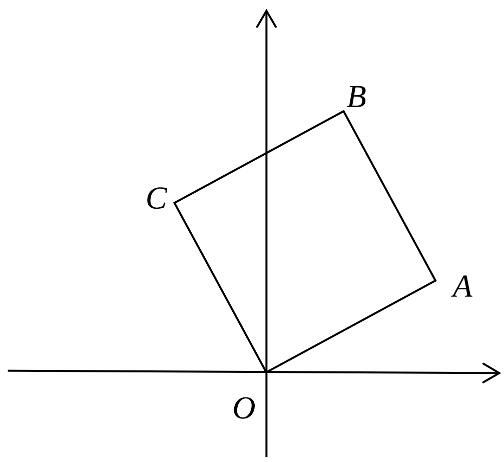
$$z^2 - (3 - 2i)z + (5 - i) = 0$$ **1**

(ii) Find the other root. **1**

(d) *OABC* is a square in the Argand diagram. **3**

B represents the complex number $2 + 2i$.

Find the complex numbers represented by *A* and *C*.



Question 11 (Continued)

(e) From $\{1,2,3,4,5,6,7,8,9\}$ codes of three digits are formed, where no digit is repeated.

(i) Find the number of possible different codes. **1**

(ii) How many of these are *not* in decreasing order of magnitude, reading from left to right? **2**

(f) Given that $\alpha, \beta,$ and γ are the roots of $x^3 - 7x + 6 = 0$, evaluate **2**

$$\alpha^3 + \beta^3 + \gamma^3$$

Question 12. (15 marks) **(Start a new answer booklet.)**

Marks

(a) Find $\int xe^{4x}\,dx$. **2**

(b) Evaluate $\int_{\frac{3\pi}{2}}^{\frac{5\pi}{2}} \frac{dx}{\cos x + 2}$. **2**

(c) Find $\int \frac{du}{8+u^3}$. **2**

(d) Evaluate $\int_0^{\frac{\pi}{4}} \cos^5\theta\, d\theta$ **2**

(e) (i) Find $\int \frac{dx}{x^2+2x+10}$. **1**

(ii) Hence find $\int \frac{x^2}{x^2+2x+10}\,dx$. **2**

(f) Consider the curve defined by $2x^2 + xy - y^2 = 0$. **2**

Find the values of $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ at the point $P(2,4)$.

(g) Sketch the locus $|z-1| + |z+1| = 4$(where *z* is a complex number), showing *x* and *y* intercepts. **2**

Question 13. (15 marks) **(Start a new answer booklet.)**

Marks

(a) Find the values of the real numbers *p* and *q* given that **2**

$$x^3+2x^2-15x-36=(x+p)^2(x+q)$$

(b) An ellipse has equation

$$\frac{x^2}{9}+\frac{y^2}{4}=1$$

(i) Find the eccentricity of the ellipse. **1**

(ii) Sketch the ellipse showing foci, directrices and intercepts. **2**

(iii) Prove that the equation of the tangent to the ellipse at the point $P(3\cos\theta, 2\sin\theta)$ is $2x\cos\theta+3y\sin\theta=6$. **3**

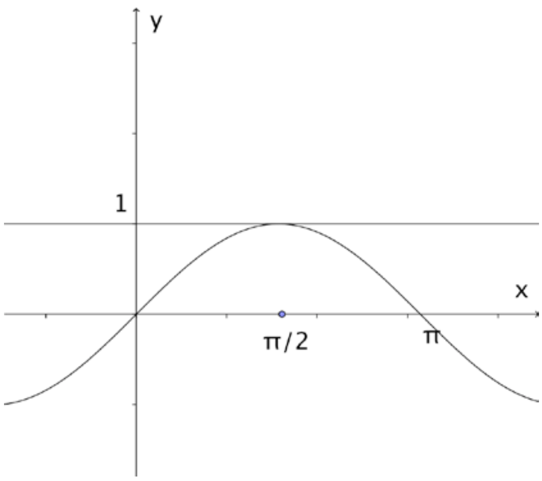
(iv) The ellipse meets the *y*-axis at points *A* and *B*. The tangents to the ellipse at *A* and *B* meet the tangent at *P*, at points *C* and *D* respectively. **3**

Prove that $AC\times BD=9$.

(c) The area defined by $0\le x\le\frac{\pi}{2}$, $0\le y\le 1$ and $y\ge \sin x$ is rotated about the line $y=1$. **4**

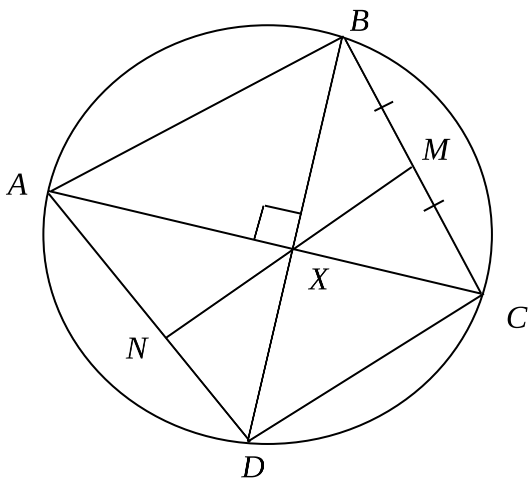
(i) Copy the diagram and shade the defined area.

(ii) Find the volume of the solid by taking slices perpendicular to the axis of rotation.



Question 14 (15 marks) **(Start a new answer booklet.)**

(a) *ABCD* is a cyclic quadrilateral. The diagonals *AC* and *BD* intersect at right-angles at *X*. *M* is the mid-point of *BC*, and *MX* produced meets *AD* at *N*.



(i) Copy the diagram to your answer booklet, then show that $BM = MX$. **1**

(ii) Show that $\angle MBX = \angle MXB$. **1**

(iii) Show that *MN* is perpendicular to *AD*. **3**

(b) The base of a solid is in the shape of an ellipse with equation $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.

Vertical cross-sections taken perpendicular to the major axis are rectangles where length is double the height.

(i) Show that the volume of a typical rectangular slice is **2**

$$\delta V = \frac{2b^2}{a^2}\left(a^2 - x^2\right).\delta x$$

(where δx is the width of the slice.)

(ii) Find the volume of the solid by integration. **2**

Question 14 (Continued)

(c) In each of the following parts, $x, y, z, w, a, b, c, d > 0$:

(i) Show that $(x+y)^2 \geq 4xy$. **1**

(ii) Show that $\left[(x+y)(z+w)\right]^2 \geq 16xyzw$ **1**

(iii) Deduce that $\dfrac{x+y+z+w}{4} \geq \sqrt[4]{xyzw}$ **2**

(iv) Hence show that (using (iii)): **2**

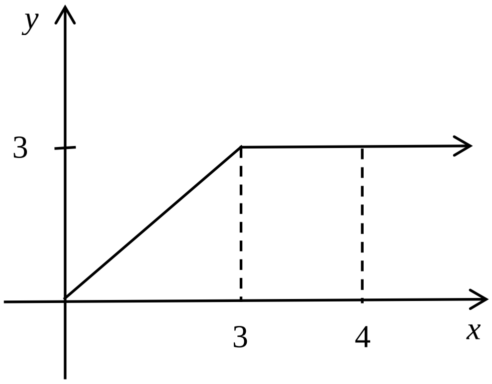
$$\frac{a}{b}+\frac{b}{c}+\frac{c}{d}+\frac{d}{a} \geq 4$$

Question 15 (15 marks) **(Start a new answer booklet.)**

Marks

(a) The graph of $y = f(x)$ is shown.

Sketch the following on separate diagrams:



(i) $y = f(4-x)$. **1**

(ii) $y = f(|x|)$. **1**

(iii) $y \times f(x) = 1$. **1**

(iv) $y^2 = f(x)$. **1**

(b) Let w be a non-real cube root of unity.

(i) Show that $1 + w + w^2 = 0$. **1**

(ii) Simplify $(1+w)^2$. **1**

(iii) Show that $(1+w)^3 = -1$. **1**

(iv) Using part (iii) simplify $(1+w)^{3n}$ where $n \in Z^+$. **1**

(v) Show that **3**

$$\binom{3n}{0} - \frac{1}{2}\left[\binom{3n}{1} + \binom{3n}{2}\right] + \binom{3n}{3} - \frac{1}{2}\left[\binom{3n}{4} + \binom{3n}{5}\right] + \binom{3n}{6} - \cdots$$

$$\cdots + \binom{3n}{3n} = (-1)^n$$

[Hint: You may use $\mathrm{Re}(w) = \mathrm{Re}(w^2) = -\frac{1}{2}$]

(c) (i) Show that $\ln(ex) > e^{-x}$ for $x \geq 1$. (Use a diagram.) **1**

(ii) Hence show that $\ln(e^n \times n!) > \dfrac{e^n - 1}{e^n(e-1)}$. **3**

Question 16 (15 marks) **(Start a new answer booklet.)**

(a) A Particle *P* of unit mass is thrown vertically downwards in a medium where the resistive force is proportional to the velocity.

(i) Taking *downwards as positive*, show that $\ddot{x} = g - kv$ for some $k > 0$. **1**

(ii) Given that the initial speed is *U* and the particle is thrown from a point *T*, distant *d* units above a fixed point *O*, (taken as the Origin) so that the initial conditions are $v = U$ and $x = -d$. **2**

Show that $v = \frac{g}{k} - \left(\frac{g-kU}{k}\right)e^{-kt}$.

(iii) Hence show that: **2**

$$x = \frac{gt - kd}{k} + \left(\frac{g-kU}{k^2}\right)\left(e^{-kt} - 1\right)$$

(iv) A second identical particle *Q* is dropped from *O*, at then same instant that *P* is thrown down. Use the above results to find expressions for *v* and *x* as functions of *t*, for the particle *Q*. **2**

(v) The particles collide. Find when this occurs, and find the speed at which they collide **3**

(b) (i) Show that: **1**

$$\sin(2r+1)\theta - \sin(2r-1)\theta = 2\sin\theta\cos 2r\theta$$

(ii) Hence shown that: **2**

$$\sin\theta\sum_{r=1}^{n}\cos 2r\theta = \frac{1}{2}\left\{\sin(2n+1)\theta - \sin\theta\right\}.$$

(iii) Hence evaluate: **2**

$$\sum_{r=1}^{100}\cos^2\frac{r\pi}{100}.$$

This is the end of the paper.

STANDARD INTEGRALS

$$\int x^n dx = \frac{1}{n+1}x^{n+1},\ n \neq -1;\ x \neq 0, \text{if } n < 0$$

$$\int \frac{1}{x} dx = \ln x,\ x > 0$$

$$\int e^{ax} dx = \frac{1}{a}e^{ax},\ a \neq 0$$

$$\int \cos ax\, dx = \frac{1}{a}\sin ax,\ \ a \neq 0$$

$$\int \sin ax\, dx = -\frac{1}{a}\cos ax,\ \ a \neq 0$$

$$\int \sec^2 ax\, dx = \frac{1}{a}\tan ax,$$

$$\int \sec ax \tan ax\, dx = \frac{1}{a}\sec ax,\ \ a \neq 0$$

$$\int \frac{1}{a^2 + x^2} dx = \frac{1}{a}\tan^{-1}\frac{x}{a},\ \ a \neq 0$$

$$\int \frac{1}{\sqrt{a^2 - x^2}} dx = \sin^{-1}\frac{x}{a},\ \ a > 0,\ \ -a < x < a$$

$$\int \frac{1}{\sqrt{x^2 - a^2}} dx = \ln\left(x + \sqrt{x^2 - a^2}\right)\ \ x > a > 0$$

$$\int \frac{1}{\sqrt{x^2 + a^2}} dx = \ln\left(x + \sqrt{x^2 + a^2}\right)$$

NOTE: $\ln x = \log_e x,\ \ x > 0$