SYDNEY BOYS' HIGH SCHOOL

MOORE PARK, SURRY HILLS

AUGUST 1995 TRIAL HSC

MATHEMATICS

4 UNIT

Time allowed - Three hours
(Plus 5 minutes reading time)

Examiners: C Kourtesis
P. Harnett

DIRECTIONS TO CANDIDATES

- ALL questions may be attempted.
- ALL questions are of equal value.
- All necessary working should be shown in every question. Full marks may not be awarded for careless or badly arranged work.
- Approved calculators may be used.
- Each section attempted is to be returned in a separate bundle, clearly showing your name. Start each question on a new page.
- If required, additional paper may be obtained from the Examination Supervisor upon request.
- This is a trial paper and does not necessarily reflect the format or content of the HSC examination fro this subject.

SECTION A (Hand up separately)

Question 1 (Start a separate page)

2 a) Find $\displaystyle\int_0^{\frac{1}{3}} \frac{dx}{1 + 9x^2}$

3 b) Find $\displaystyle\int \sin^{-1}x \, dx$

2 c) Find $\displaystyle\int \frac{\cos\sqrt{x}}{\sqrt{x}} \, dx$

4 d) (i) If $x = \sec\theta$ show that

$$\frac{dx}{d\theta} = \sec\theta\tan\theta$$

$x = \frac{1}{\cos\theta}$ $(\cos\theta)^{-1}$

$-1(\cos$ —

(ii) Use the substitution $x = \sec\theta$ to find

$$\int_{\frac{2}{\sqrt{3}}}^{2} \frac{\sqrt{x^2 - 1}}{x} \, dx$$

4 e) Find $\displaystyle\int_0^1 \frac{1}{(x+1)(x^2+1)} \, dx$

$A + C = 1$

$\frac{1}{2}$

Question 2 (**Start a separate page**)

2 a) If $z = \cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}$

find z^4 in the form $x + iy$, where x and y are real.

3 b) Sketch the region in the number plane which simultaneously satisfies

$$\frac{\pi}{4} \le \arg(z - i) \le \frac{3\pi}{4}$$

and

$$|z - i| \le 2$$

6 c) A represents the point, $\omega = -2i$. B and C represent the complex numbers ρ and α, which are the square roots of ω.

(i) Given that $\text{Im}(\rho) > 0$, plot A, B, and C on an Argand diagram.

5

(ii) Find ρ and α in the form $x + iy$, where x and y are real.

(iii) D represents the complex number τ. If ABCD is a parallelogram, find τ in the form $x + iy$.

4 d) (i) Find the two complex numbers z_1, z_2 which satisfy

$$3z\bar{z} + 2(z - \bar{z}) = 39 + 12i$$

(ii) On an Argand diagram these numbers are represented by the points P and Q. R represents the complex number i. Show that angle PRQ is a right angle.

2

$x^2+y^2=0$

$2xy=-2$.

$x^2-y^2=-4$.

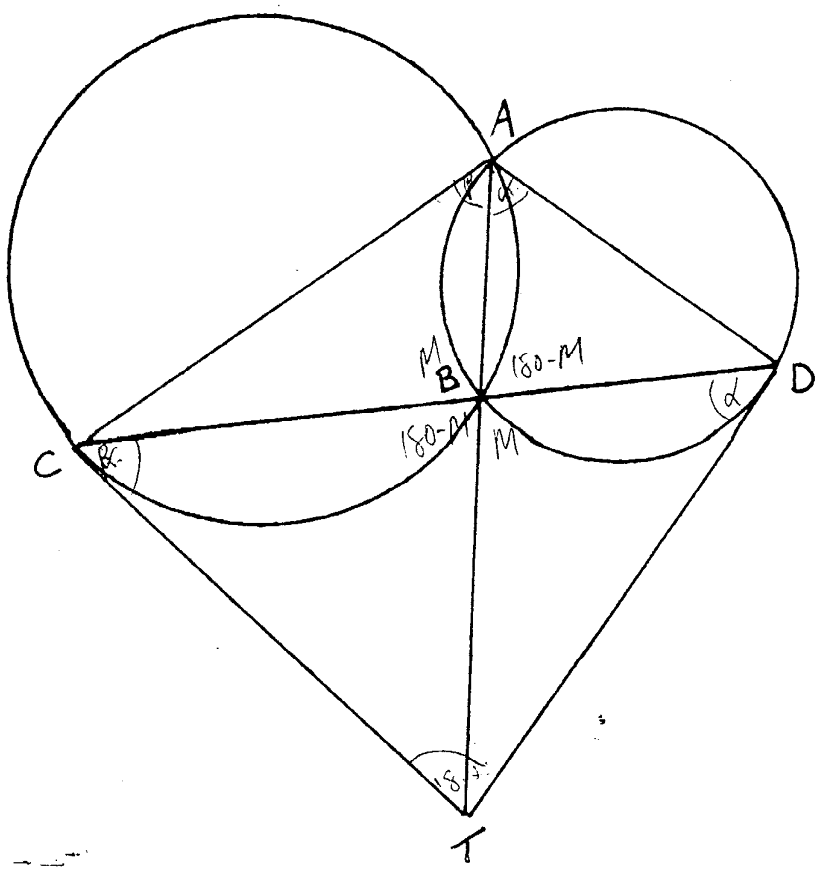
$(x^2-y^2)^2=(x^2+y^2)^2 - \cancel{4xy}\ (2xy)^2$

$0^2 - (\alpha)$.

SECTION B (Hand up separately)

Question 3 (Start a separate page)

5 a)



BAC, BAD are two circles such that the tangents at C and D meet at T on AB produced. If CBD is a straight line, prove that:

3 (i) TCAD is a cyclic quadrilateral.

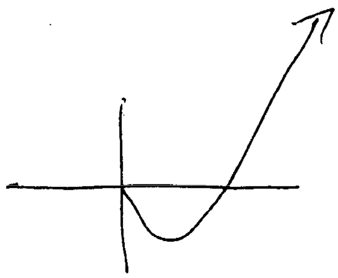
1 (ii) ∠TAC = ∠TAD.

1 (iii) TC = TD.

Question 3 (continued)

10 b) Consider the function

$$y = x^{\frac{1}{4}}(x - 10)$$



1 (i) Find the roots of the function.

1 (ii) State the natural domain of the function.

1 (iii) Find $\frac{dy}{dx}$

3 (iv) Find any stationary points and determine their nature.

1 (v) What happens to y as $x \to \infty$?

1 (vi) Describe the nature of the function near $x = 0$.

2 (vii) Sketch the curve.

Question 4 (Start a separate page)

5 a) (i) On the same number plane sketch the graphs of

$$y = |x| - 3 \quad \text{and} \quad y = 5 + 4x - x^2$$

(ii) Hence, or otherwise, solve

$(5-x)(1+x)$

$$\frac{|x| - 3}{5 + 4x - x^2} > 0$$

2 b) Given the polynomial P(x) where

$$P(x) = nx^{n+1} - (n+1)x^n + 1, \qquad (n \neq 0)$$

Prove that P(x) is divisible by $(x - 1)^2$

4 c) The equation $x^3 + 2x - 1 = 0$ has roots α, β, γ. Find:

(i) the value of $\alpha^2 + \beta^2 + \gamma^2$

(ii) the equation with roots $-\alpha, -\beta, -\gamma$.

4 d) P(x) is a polynomial with the following form:

$$P(x) = Ax^3 + Bx^2 + Cx + D,$$

where A, B, C, D are real.
P(x) has roots 5 and i. When divided by (x - 2) the remainder is 3.
Find P(x).

2

Question 5 (Start a separate page)

3 a) On a given city block there are 10 houses on one side of the street and 10 on the other. In how many ways can seven trees be planted each one in front of a different house:

^{10}C

(i) If they must all be on the same side of the street?

10 ~~...~~
10 house
7 trees.

(ii) If there must be four trees on one side of the street and three on the other?

$2 \times {}^{10}C_3 \times {}^{10}C_4$

6 b) **1** (i) Sketch the graph of the function $y = e^{kx}$ where **k** is a positive constant.

$y = ke^{kx}\, x$

2 (ii) Find the equation of the tangent to the curve $y = e^{kx}$ which passes through the origin.

3 (iii) Hence, or otherwise, find the number of real roots of the equation:

$$e^{kx} = x$$

if

2

α) $k = \frac{1}{e}$

β) $k < \frac{1}{e}$

γ) $k > \frac{1}{e}$

6 c) The region bounded by the graphs of y = lnx, y = 1, and x = 3 is rotated about the y-axis.

1 (i) Show this region on a clear diagram. $1 = \ln x.$ $x =$

5 (ii) Use the method of cylindrical shells to prove that the volume of the resulting solid is

$$(9\pi in3 - \frac{27}{2}\pi + \frac{\pi}{2}e^2) \text{ units}^3$$

SECTION C (Hand up separately)

Question 6 (**Start a separate page**)

The only force acting on a particle, moving in a straight line, is a resistance $\mathbf{mk(c^2 + v^2)}$ acting in that line, where **m** is the mass of the particle in kilograms, **v** is the velocity in metres per second and **k, c** are positive constants.
If the distance moved is **x** metres:

1 a) Show that the acceleration, $\ddot{x}$, of the particle is given by

$$\ddot{x} = v\frac{dv}{dx}$$

$a = \frac{dv}{dt}$ $\frac{dv}{dx} \cdot \frac{dx}{dt}$

2 b) Show that the equation of motion is given by

$$\frac{dv}{dx} = \frac{-k(c^2 + v^2)}{v}$$

3 c) If the particle starts to move with velocity **U** (> 0), show that:

$$x = \frac{1}{2k}\ln[\frac{c^2 + U^2}{c^2 + v^2}]$$

9 d) The particle comes to rest in a distance S metres. Its speed is $\frac{1}{3}$U when it has moved a distance of $\frac{1}{2}$**S** metres.

4 (i) Show that:

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$$63c^2 = U^2$$

3 (ii) If the particle has moved a distance of **x** metres show that

2

$$64U^2e^{-2kx} - 63v^2 = U^2$$

2 (iii) Show that

$$0 \leq S \leq \frac{3}{k}\ln 2.$$

2

Question 7 **(Start a separate page)**

The points P_1, P_2, P_3, P_4 with parameters t_1, t_2, t_3, t_4 respectively lie on the rectangular hyperbola:

$$x = ct$$
$$y = \frac{c}{t}$$

The tangent at P_1 cuts the x-axis at A and the y-axis at B. The tangent at P_2 is parallel to the tangent at P_1 and cuts the x-axis at C and the y-axis at D.

2 a) Show that the equation of the tangent at P_1 is:

$$x + t_1^2 y = 2ct_1$$

2 b) Find the coordinates of A, B, C, D.

3 c) 2 (i) Prove that ABCD is a rhombus.

1 (ii) Show that the area of the rhombus is constant.

2 d) Show that the chord P_1P_2 has equation:

$$x + t_1 t_2 y = c(t_1 + t_2)$$

3 e) The circle $x^2 + y^2 = R^2$ passes through the points P_1, P_2, P_3, P_4. Prove that:

$$t_1 t_2 t_3 t_4 = 1$$

3 f) Prove that if the chord P_1P_2 passes through the origin then P_3P_4 is a diameter of the circle.

Question 8 (Start a separate page)

5 (a) A triangle ABC lies on a horizontal plane. An object P vertically above A is viewed from three different points B, M and C which are in line and on the same horizontal plane as triangle ABC. If M is halfway between B and C where BC = 50 metres and the angles of elevation from B, M and C to P are 60°, 45° and 30° respectively. Prove that the height of the object P above A is

3

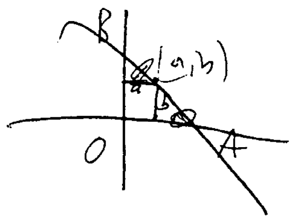
$$\frac{25\sqrt{6}}{2} \text{ metres}$$

6 b) A straight line is drawn through a fixed point (a,b) in the first quadrant to cut the positive parts of the x- and y- axes at A and B respectively. If angle OAB is θ (where O is the origin).

6

2 (i) Prove that the length of AB is given by:

$$AB = a\sec\theta + b\operatorname{cosec}\theta$$



2 (ii) Show that the length of AB will be a minimum if

$$\cot\theta = \left(\frac{a}{b}\right)^{\frac{1}{3}}$$

2 (iii) Show the minimum length of AB is $\left(a^{\frac{2}{3}} + b^{\frac{2}{3}}\right)^{\frac{3}{2}}$

4 c) If $\frac{1}{x_1} + \frac{1}{x_2} + \frac{1}{x_3} + \dots + \frac{1}{x_{n-1}} + \frac{1}{x_n}$ is an arithmetic series. Show that:

3

$$x_1x_2 + x_2x_3 + x_3x_4 + \dots + x_{n-1}x_n = (n-1)x_1x_n$$

END OF PAPER