

# SYDNEY BOYS' HIGH SCHOOL

MOORE PARK, SURRY HILLS



**TRIAL HSC**  
**August 1999**

# MATHEMATICS

**4 UNIT ADDITIONAL**

*Time allowed — 3 Hours  
(plus 5 minutes reading time)*

*Examiner: E. Choy & P.S. Parker*

## **DIRECTIONS TO CANDIDATES**

- *ALL* questions may be attempted.
- All necessary working should be shown in every question. Full marks may not be awarded for careless or badly arranged work.
- Approved calculators may be used.
- Use a new booklet for each question.
- If required, additional booklets may be obtained from the Examination Supervisor upon request.

This is a trial paper and does not necessarily reflect the format or content of the HSC examination for this subject.

Question 1. (Start a new booklet)

Marks

(a) Evaluate

2

$$\int_1^3 \frac{x \, dx}{\sqrt{10-x^2}}$$

(b) (i) Evaluate

4

$$\int \frac{dx}{x(x+1)} \text{ for } x > 0$$

(ii) Hence, using the substitution  $u = e^x$  evaluate

$$\int \frac{dx}{1+e^x}$$

(c) Evaluate

2

$$\int_1^{e^2} x \ln \sqrt{x} \, dx$$

(d) Evaluate

3

$$\int \sin^2 x \cos^5 x \, dx$$

(e) Evaluate

4

$$\int \frac{\cos x \, dx}{\sin x + \sin^2 x}$$

Question 2. (Start a new booklet)

Marks

(a) (i)  $\frac{3+2i}{4-i} = a+ib$ , where  $a$  and  $b$  are real.

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Find  $a$  and  $b$ .

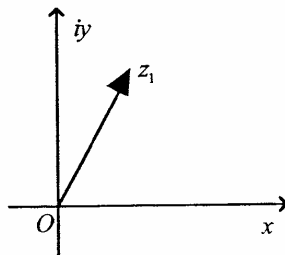
- (ii) If  $z = 4 + 3i$ , express  $z$  in mod-arg form (give argument to the nearest degree) and write down the value of  $\arg(z^5)$ .

- (b) (i) Sketch the set of points in the complex plane satisfying both inequalities. Show all intercepts with the real and imaginary axes. 6

$$|6 - 5i + z| \leq 5 \text{ and } \text{Im}(z) \leq 5$$

- (ii) Sketch the locus of  $\text{Re}(z\bar{z} - 2z + 8\bar{z} - 16) = 0$ , showing all intercepts with the real and imaginary axes.

(c)



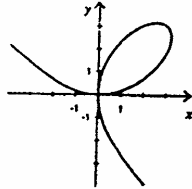
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The diagram above shows a vector representing the complex number  $z_1$ . Copy the diagram and on it show vectors representing the complex numbers  $iz_1$  and  $z_1 - iz_1$ . Hence sketch a rectangle labelling the vertices in terms of  $z_1$ .

Question 3. (Start a new booklet)

Marks

- (a) The curve below is called the *folium of Descartes* and its equation is  $x^3 + y^3 = 6xy$  3



- (i) Find  $\frac{dy}{dx}$
- (ii) Find the equation of the tangent line to the folium of Descartes at the point (3, 3).
- (b) Sketch  $f(x) = \frac{x-2}{x-3}$  and hence sketch on separate number planes: 9
- (i)  $y = |f(x)|$
- (ii)  $y = [f(x)]^2$
- (iii)  $y = \frac{1}{f(x)}$
- (iv)  $y = \frac{x-2}{|x-3|}$
- (c) Sketch  $(1+x^2)y^2 = x$ , making sure to indicate turning points and intercepts on your diagram. 3

Question 4. (Start a new booklet)

Marks

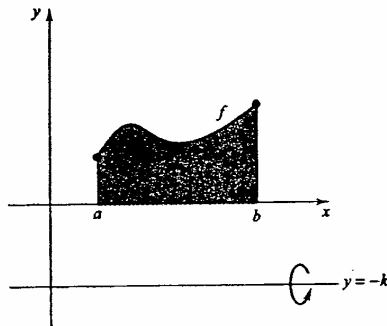
- (a) (i) Use the factor theorem to factorise the polynomial  $f(x) = x^4 - 4x^3 - 9x^2 + 16x + 20$  7
- (ii) Without any use of calculus, sketch  $y = x^4 - 4x^3 - 9x^2 + 16x + 20$ , showing all  $x$  and  $y$  intercepts.
- (iii) The polynomial  $x^4 - 4x^3 - 9x^2 + (16 - m)x + 20 - b$  has two double roots  $\alpha$  and  $\beta$ .
- (\alpha) Using the sum and products of roots, write down four equations involving  $\alpha$ ,  $\beta$ ,  $m$  and  $b$ .
- (\beta) Hence find  $\alpha$ ,  $\beta$ ,  $m$  and  $b$ .
- (iv) Hence or otherwise determine the equation of the unique line which is tangent at two distinct points to the curve  $y = x^4 - 4x^3 - 9x^2 + 16x + 20$ .
- (b) Let  $\alpha$  be the complex root of the equation  $z^7 = 1$  with smallest positive argument. 8  
 Let  $\theta = \alpha + \alpha^2 + \alpha^4$  and  $\phi = \alpha^3 + \alpha^5 + \alpha^6$ .
- (i) Explain why  $\alpha^7 = 1$  and  $1 + \alpha + \alpha^2 + \alpha^3 + \alpha^4 + \alpha^5 + \alpha^6 = 0$
- (ii) Show that  $\theta + \phi = -1$  and  $\theta\phi = 2$ , and hence write down a quadratic equation whose roots are  $\theta$  and  $\phi$ .
- (iii) Show that  $\theta = -\frac{1}{2} + \frac{i\sqrt{7}}{2}$  and  $\phi = -\frac{1}{2} - \frac{i\sqrt{7}}{2}$
- (iv) Write down  $\alpha$  in mod-arg form, and show that:
- (\alpha)  $\cos\frac{4\pi}{7} + \cos\frac{2\pi}{7} - \cos\frac{\pi}{7} = -\frac{1}{2}$ ;
- (\beta)  $\sin\frac{4\pi}{7} + \sin\frac{2\pi}{7} - \sin\frac{\pi}{7} = \frac{\sqrt{7}}{2}$

Question 5. (Start a new booklet)

Marks

- (a) In the diagram below,  $y = f(x)$  is a continuous, nonnegative function on  $a \leq x \leq b$  and let  $\Omega$  be the region between the graph of  $y = f(x)$  and the  $x$  axis. Let  $k$  be a positive constant. Prove that if the region  $\Omega$  is revolved around the line  $y = -k$ , then the volume of the solid of revolution is given by:

$$V = \pi \int_a^b ([f(x) + k]^2 - k^2) dx$$



- (b) The only force acting on a particle moving in a straight line is a resistance,  $mk(c + v)$ , acting in that line, where  $m$  is the mass of the particle,  $v$  its velocity and  $k, c$  are positive constants. The particle had an initial velocity  $U (> 0)$  and comes to rest in time  $T$ . Its velocity is  $\frac{1}{4}U$  at time  $\frac{1}{2}T$ .

- (i) Show that  $\frac{c + v}{c + U} = e^{-kt}$
- (ii) Hence show that  $c = \frac{1}{8}U$
- (iii) Show also that at time  $t$ ,  $\frac{8v}{U} = 9e^{-kt} - 1$

**Question 6. (Start a new booklet)****Marks**

- (a) Consider the geometric series

**7**

$$S = 1 + (1+x) + (1+x)^2 + \dots + (1+x)^n$$

- (i) Show that

$$S = \frac{(1+x)^{n+1} - 1}{x}$$

- (ii) Hence show that

$$S = \binom{n+1}{1} + \binom{n+1}{2}x + \dots + \binom{n+1}{r+1}x^r + \dots + \binom{n+1}{n+1}x^n$$

- (iii) Hence prove

$$\binom{n}{r} + \binom{n-1}{r} + \binom{n-2}{r} + \dots + \binom{r}{r} = \binom{n+1}{r+1}$$

- (b) In how many ways can we distribute 21 books among 3 people, Martin, Barton and Hugo, so that Martin and Barton together receive twice as many books as Hugo? **4**
- (c) In how many ways can the 7 letters  $a b X X X c d$  be arranged in a line if no two  $X$ 's are together? **4**

Question 7. (Start a new booklet)

Marks

(a) Show that

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$$\int_0^1 (1-x^2)^n dx = \frac{2 \times 4 \times 6 \times 8 \times \dots \times 2n}{3 \times 5 \times 7 \times \dots \times (2n+1)}, n \geq 0$$

(b) (i) Show that  $p_3(x) = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!}$  has at least one real root.

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(ii) Show that  $p_2(x) = 1 + x + \frac{x^2}{2!}$  has no real roots and deduce that  $p_3(x)$  has exactly one real root.

(iii) Deduce that  $p_4(x) = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} > 0$  for all  $x$ .

(iv) Let  $p_n(x) = \sum_{k=0}^n \frac{x^k}{k!}$ ,  $n \geq 1$ .

Use induction to prove that if  $n$  is even then  $p_n(x) > 0$  for all  $x$  and if  $n$  is odd then  $p_n(x)$  has exactly one real root and this root is negative.

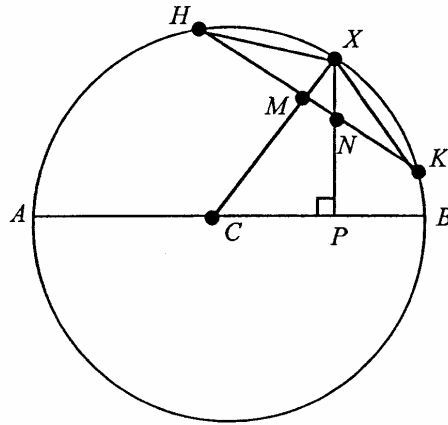


Question 8. (Start a new booklet)

Marks

(a)

6



$AB$  is a diameter of a circle, whose centre is  $C$ , and  $X$  is any point on the circumference.  
 $P$  is the foot of the perpendicular drawn from  $X$  to  $AB$ .  
 $H$  and  $K$  are points on the circumference such that  $XH = XK$ .  
 $HK$  meets  $XP$  in  $N$  and  $XC$  in  $M$ .  
 Prove that the points  $M, N, C$  and  $P$  are concyclic.

**Question 8. (continued)**

**Marks**

- (b) A bowl shaped like a hemisphere of radius 8 cm contains a scoop of ice cream (a sphere) with a diameter of 4 cm.

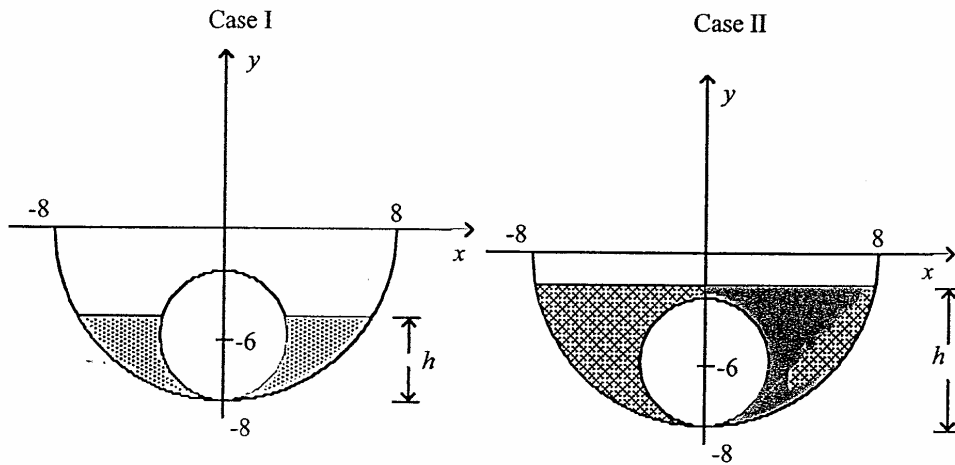
**9**

If the bowl is filled with milk to a depth of  $h$  cm, what is the volume of milk it contains?

You will need to consider two cases:

- (I) The case where the ice cream is partially submerged.  
(II) The case where the ice cream is totally submerged.

You can assume in both cases that the ice cream doesn't melt or float.



**THIS IS THE END OF THE PAPER.**