SYDNEY BOYS' HIGH SCHOOL

MOORE PARK, SURRY HILLS

TRIAL HIGHER SCHOOL CERTIFICATE EXAMINATION 2000

MATHEMATICS

4 UNIT ADDITIONAL

Time allowed — 3 Hours
(plus 5 minutes reading time)

Examiner: C. Kourtesis

DIRECTIONS TO CANDIDATES

- *ALL* questions may be attempted.
- All necessary working should be shown in every question. Full marks may not be awarded for careless or badly arranged work.
- Approved calculators may be used.
- Start **each** section in a new booklet. Section A (questions 1, 2, 3), Section B (questions 4, 5, 6) and Section C (questions 7, 8).
- If required, additional booklets may be obtained from the Examination Supervisor upon request.

This is a trial paper and does not necessarily reflect the format or content of the HSC examination for this subject.

SECTION A

Question 1. (Start a new booklet) **15 Marks**

(a) If $z = (1-i)^{-1}$ 5

(i) Express $\bar{z}$ in modulus-argument form,

(ii) If $(\bar{z})^{13} = a + ib$ where a and b are real numbers, find the values of a and b.

(b) Find the cartesian equation of the locus of a point P which represents the complex number z where 2

$$|z-i| = |z|$$

(c) Sketch the region in the complex plane where 3

$$\text{Re}\left[(2-3i)z\right] < 12$$

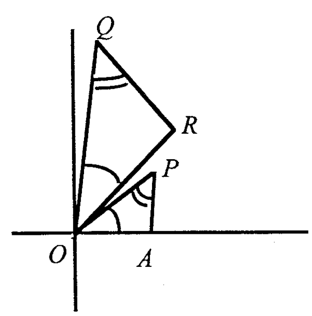
(d) (i) On an Argand diagram sketch the locus of a point P, corresponding to the complex number z, where 3

$$|z-3| = 3$$

(ii) Use your diagram in (i) to explain why

$$\arg(z-3) = \arg z^2$$

(e) 2



The points A, P and R in the complex plane correspond to the complex numbers 1, $\frac{3}{2} + i$ and $2 + 2i$ respectively. Triangles OAP and ORQ are similar with corresponding angles as indicated.

Find the complex number represented by Q.

Question 2. 15 Marks

(a) Find $\int \frac{dx}{x^2-4x+9}$ 2

(b) (i) Express $\frac{4x-2}{(x^2-1)(x-2)}$ in the form $\frac{Ax+B}{x^2-1}+\frac{C}{x-2}$, where A, B and C are constants. 5

(ii) Hence evaluate

$$\int_3^6 \frac{4x-2}{(x^2-1)(x-2)}\,dx$$

(c) Find $\int \frac{e^{2x}}{e^x-1}\,dx$ by using the substitution $u=e^x$. 3

(d) (i) If $u_n=\int_0^{\frac{\pi}{2}} \theta \sin^n\theta \, d\theta$ where $n \ge 1$, use integration by parts to prove that 5

$$u_n=\frac{n-1}{n}u_{n-2}+\frac{1}{n^2}$$

(ii) Hence show that $u_5=\frac{149}{225}$

Question 3. **15 Marks**

(a) The polynomial $P(z)$ has equation **3**

$$P(z) = z^4 - 2z^3 - z^2 + 2z + 10$$

Given that $z - 2 + i$ is a factor of $P(z)$, express $P(z)$ as a product of two real quadratic factors.

(b) The remainder when $x^4 + ax + b$ is divided by $(x-2)(x+1)$ is $x+2$. Find the values of a and b. **2**

(c) (i) Show that **10**

$$\tan 3\theta = \frac{3\tan\theta - \tan^3\theta}{1 - 3\tan^2\theta}$$

(ii) Find the general solution of the equation $\tan 3\theta = \sqrt{3}$

(iii) Using the substitution $x = \tan\theta$, express the equation in (ii) as a polynomial equation in terms of x.

(iv) Hence show that $\tan\frac{\pi}{9} + \tan\frac{4\pi}{9} + \tan\frac{7\pi}{9} = 3\sqrt{3}$

(v) Find the polynomial of least degree that has zeros

$$\left(\cot\frac{\pi}{9}\right)^2, \left(\cot\frac{4\pi}{9}\right)^2, \left(\cot\frac{7\pi}{9}\right)^2$$

SECTION B

Question 4. (Start a new booklet) **15 Marks**

(a) Consider the function 9

$$F(x)=\left(\frac{x+4}{x}\right)^2,\ x\neq 0$$

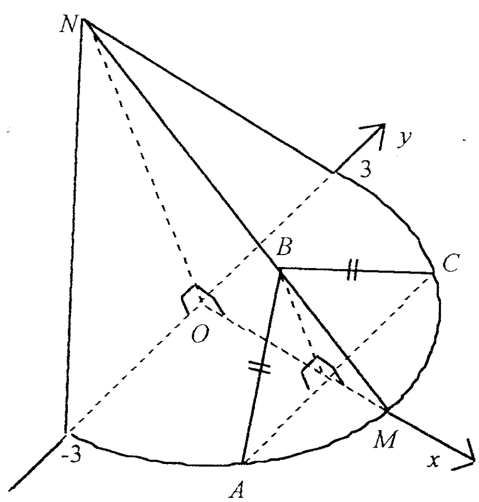
(i) Find all the turning points of $y=F(x)$,

(i) Determine the coordinates of the point of inflexion,

(iii) Find the equations of any asymptotes,

(iv) Sketch the curve $y=F(x)$ for all points in its domain.

(b) 6



A solid figure has a semi circular base of radius 3 cm. Cross sections taken perpendicular to the x axis are isosceles triangles. The vertical cross section containing the radius OM of the base of the solid is a right isosceles triangle OMN, where $OM = ON$.

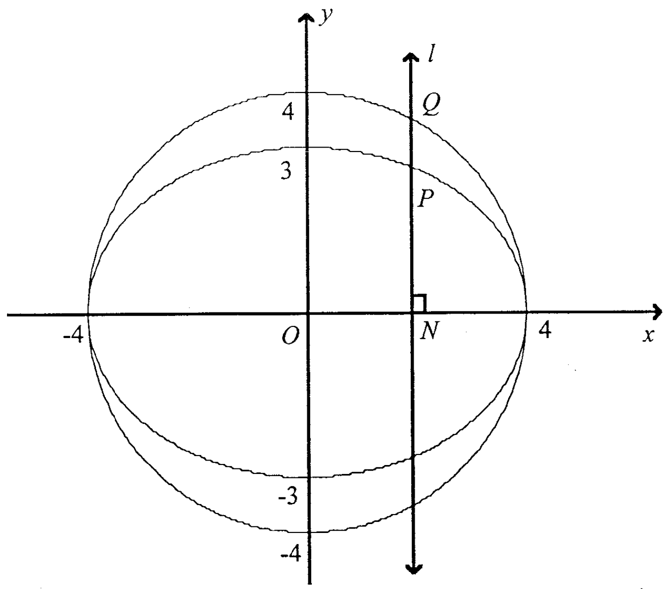
(i) Show that the area, A, of triangle ABC (where $AB = BC$) is given by

$$A = (3-x)(9-x^2)^{\frac{1}{2}}$$

(ii) Find the volume of the solid.

Question 5. 15 Marks

(a) 13



The diagram shows the ellipse, E, with equation $\frac{x^2}{16}+\frac{y^2}{9}=1$ and its auxiliary circle C. The coordinates of a point P on E are $(4\cos\theta, 3\sin\theta)$.

A straight line, l, parallel to the y axis intersects the x axis at N and the curves E and C at the points P and Q respectively.

(i) Find the eccentricity of E,

(ii) Write down the coordinates of N and Q,

(iii) Find the equations of the tangents at P and Q to the curves E and C respectively,

(iv) The tangents at P and Q intersect at a point R. Show that R lies on the x axis,

(v) Prove that $ON.OR$ is independent of the positions of P and Q.

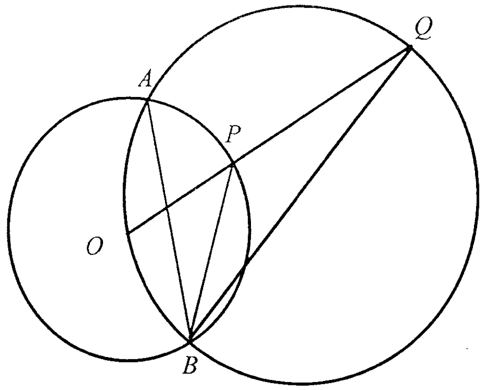
(b) State whether the following is True or False. Give brief reasons. 2

Note: You are NOT required to find the primitive function.

$$\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \tan^9\theta \, d\theta > 0$$

Question 6. **15 Marks**

(a) **5**



In the diagram above, the centre O of the small circle APB lies on the circumference of the larger circle AQB. The points O, P and Q are collinear.

Prove that BP bisects $\angle ABQ$

(b) (i) Sketch the region in the number plane that contains all points satisfying simultaneously the inequalities **6**

$$x \le 1,\ y \ge 1 \text{ and } y \le e^x$$

(ii) This region is rotated through one complete revolution about the x axis. Use the method of cylindrical shells to show that the volume of the resulting solid is

$$\frac{\pi}{2}(e^3 - 3)$$

(c) If a function $f(x)$ is continuous for $a \le x \le b$ **4**

(i) Show that $\left| \int_a^b f(x)\,dx \right| \le \int_a^b |f(x)|\,dx$

(ii) Hence prove that

$$\left| \int_0^{\pi} 4^x \cos x\,dx \right| \le \frac{2^{2\pi} - 1}{2\ln 2}$$

SECTION C

Question 7. (Start a new booklet) **15 Marks**

(a) A particle of mass m is projected vertically upwards in a medium where it experiences a resistance of magnitude mkv^2 where k is a positive constant and v is the velocity of the particle. 11

During the downward motion the terminal velocity of the particle is V. It's initial velocity of projection is $\frac{1}{3}$ of this terminal velocity.

(i) By considering the forces on the particle during its downward motion, show that

$$kV^2 = g$$

(where g is the acceleration due to gravity)

(ii) Show that during its upward motion the acceleration of the particle $\ddot{x}$ is given by

$$\ddot{x} = -g\left(1 + \frac{v^2}{V^2}\right)$$

(iii) If the distance travelled by the particle in its upward motion is x when its velocity is v, show that the maximum height H reached is given by

$$H = \frac{V^2}{2g}\ln\left(\frac{10}{9}\right)$$

(iv) The velocity of the particle is v when it is has fallen a distance y from its maximum height. Show that

$$y = \frac{V^2}{2g}\ln\left[\frac{V^2}{V^2 - v^2}\right]$$

(v) The velocity of the particle is U when it returns to its point of projection. Show that

$$\frac{V}{U} = 10^{\frac{1}{2}}$$

(b) (i) From 11 distinct consonants and 5 distinct vowels, how many words can be formed, each containing 5 distinct consonants and 3 distinct vowels? 4

(ii) In how many ways is it possible to allocate 6 people to 3 different courts in a singles tennis tournament ?

Question 8. 15 Marks

(a) (i) Show that $\frac{a+b}{2} \geq \sqrt{ab}$ for all positive numbers a and b. 4

(ii) If a, b, c and d are positive numbers prove that

$$4(ab + bc + cd + ad) \leq (a + b + c + d)^2$$

(b) If u and v are real numbers such that $u + v \neq 0$ and $v \neq 0$, 5

(i) Show that if there is only one real root of the equation $x^2 + ux + v = 0$ (where $0 < x < 1$) then

$$v(1 + u + v) < 0$$

(ii) Hence, or otherwise, prove that the equation

$$\frac{1}{x+2} + \frac{u}{x+1} + \frac{v}{x} = 0$$

has only one positive root.

(c) Given the function $f(x) = x^n e^{-x}$ where n is a positive integer and $x > 0$: 6

(i) Prove that there is only one turning point and that this occurs at $x = n$. Deduce that it is a maximum turning point.

(ii) Sketch the graph of $y = f(x)$,

(iii) By considering the values of $f(n)$, $f(n-1)$ and $f(n+1)$ prove that

$$\left(1+\frac{1}{n}\right)^n < e < \left(1-\frac{1}{n}\right)^{-n}$$

THIS IS THE END OF THE PAPER.