SYDNEY BOYS HIGH SCHOOL

MATHEMATICS EXTENSION 2

Trial Higher School Certificate 2001

Time Allowed: 3 hours (plus 5 minutes reading time)

Total Marks: 120

Examiner: Mr R Dowdell, Mr PS Parker

INSTRUCTIONS:

- Attempt *all* questions.
- *All* questions are of equal value.
- All necessary working should be shown in every question. Full marks may not be awarded if work is careless or badly arranged.
- Standard integrals are provided on the last page. Approved calculators may be used.
- Return your answers in 8 booklets, 1 for each question. Each booklet must show your name.
- If required, additional Writing Booklets may be obtained from the Examination Supervisor upon request.

NOTE: This is a trial paper only and does not necessarily reflect the content or format of the final Higher School Certificate Examination Paper for this subject.

Question 1: Marks

(a) Evaluate $\int_0^{\frac{3}{2}} \frac{dx}{\sqrt{9-x^2}}$ 2

(b) Find $\int x^3 e^{x^4+7}\, dx$ 2

(c) (i) Express $\frac{x^2+x+2}{(x^2+1)(x+1)}$ in the form $\frac{Ax+B}{x^2+1}+\frac{C}{x+1}$, where A, B and C are constants. 3

(ii) Hence find $\int \frac{x^2+x+2}{(x^2+1)(x+1)}\, dx$.

(d) Using integration by parts or otherwise, evaluate $\int_0^{\frac{1}{2}} \sin^{-1} x\, dx$ 3

(e) By using the substitution $x=\pi-y$, or otherwise, evaluate $\int_0^{\pi} x \sin^3 x\, dx$ 5

Question 2: START A NEW BOOKLET Marks

(a) $\dfrac{4+3i}{1+\sqrt{2}\,i} = a + ib$, for a, b real. 2

Find the exact values of a and b.

(b) Given $z = 1 - \sqrt{3}\,i$, 3

(i) show that z^2 is a real multiple of $\dfrac{1}{z}$;

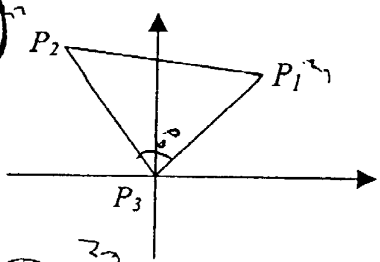
(ii) plot z, z^2, $\dfrac{1}{z}$ on an Argand diagram.

(c) Sketch the region represented by 2

$|z| \le 4$ and $\dfrac{\pi}{3} < \arg z \le \dfrac{2\pi}{3}$.

(d) (i) Show that $\dfrac{(1+i\sqrt{3})^6}{(\sqrt{3}-i)^k} = 2^{6-k}\, cis\left(\dfrac{k\pi}{6}\right)$. 4

(ii) For what values of k is $\dfrac{(1+i\sqrt{3})^6}{(\sqrt{3}-i)^k}$ purely imaginary?

(e) 

The points P_1, P_2 and P_3 represent the complex numbers z_1, z_2 and z_3 respectively. (NOTE: $z_3 = 0$.) 4

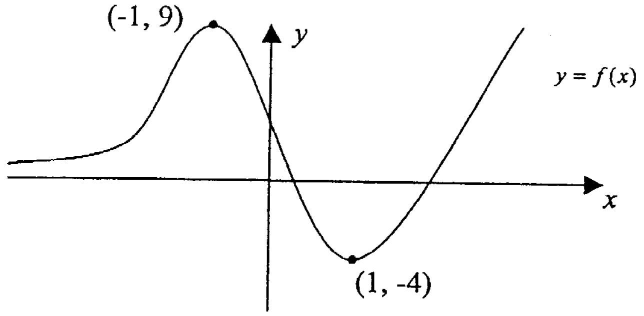
(i) If P_1, P_2 and P_3 are the vertices of an equilateral triangle, show that $\dfrac{z_2}{z_1} = \dfrac{1+i\sqrt{3}}{2}$ and deduce that $z_1^{\,2} + z_2^{\,2} = z_1 z_2$.

(ii) Deduce that if z_1, z_2 and z_3 are **ANY** three complex numbers at the vertices of an equilateral triangle then

$z_1^{\,2} + z_2^{\,2} + z_3^{\,2} = z_1 z_2 + z_2 z_3 + z_3 z_1$

Question 3: START A NEW BOOKLET Marks

(a) If the curve below represents $y = f(x)$, 12



make neat sketches, on separate axes, of

(i) $y = (f(x))^2$

(ii) $y = \dfrac{1}{f(x)}$

(iii) $y = |f(x)|$

(iv) $y = f(|x|)$

(v) $y^2 = f(x)$

(vi) $y = f'(x)$

(b) Two sides of a triangle are in the ratio 3:1 and the angles opposite these sides differ by $\dfrac{\pi}{6}$. Show that the smaller of the two angles is $\tan^{-1}\left(\dfrac{1}{6-\sqrt{3}}\right)$. 3

Question 4: START A NEW BOOKLET **Marks**

(a) $1+i$ and $3-i$ are zeroes of a real, monic polynomial, $p(x)$, of degree 4. 3

(i) Express $p(x)$ as a product of two real quadratic factors.

(ii) Explain briefly why the polynomial $p(x)$ cannot take negative values.

(b) $x^3+3px+q=0$ has a double root of $x=k$. 4

(i) Show that $p=-k^2$.

(ii) Show that $4p^3+q^2=0$.

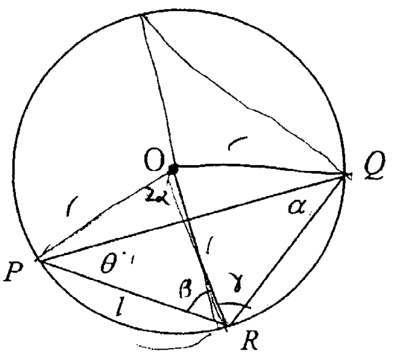
(iii) Hence factorise $x^3-6ix+4-4i$ into linear factors, given that it has a repeated factor.

(c) Consider $f(x)=x^3+9x+26$ and $g(x)=x^2+26x-27$. 3

(i) Verify that $f\left(x-\dfrac{3}{x}\right)=\dfrac{g(x^3)}{x^3}$.

(ii) Hence solve $f(x)=0$.

(d) 5



ΔPQR is a triangle inscribed in a circle of radius r. PR has length l, and $\angle PQR=\alpha$

(i) Show that $l=2r\sin\alpha$.

(ii) If $\angle QPR=\theta$, show that the area of ΔPQR is $r^2\sin\alpha(\cos\alpha-\cos(2\theta+\alpha))$

(iii) If $PQ=QR$, what is the area of ΔPQR in terms of r and α ?

Question 5: START A NEW BOOKLET

Marks

(a) A mass of m kilograms falls from rest. It experiences resistance during its fall equal to mkv where v is its speed in metres per second and k is a positive constant. Let x be the distance in metres of the mass from its starting point measured positively as it falls and t be the time in seconds. 8

(i) Show that the equation of motion of the mass is $\ddot{x} = g - kv$ where g is the acceleration due to gravity.

(ii) Show that the terminal velocity is $\dfrac{g}{k}$.

(iii) Find v as a function of t.

(iv) Find x as a function of t.

(b) (i) In how many ways can 10 students be grouped into two teams of 5 to play a game of basketball? 2

(ii) Two of the 10 students are twins. If the teams are formed at random, what is the probability that the twins play on the same team?

(c) A group of men and women is seated randomly around a circular table. What is the probability that none of the men are sitting next to each other if there are 5

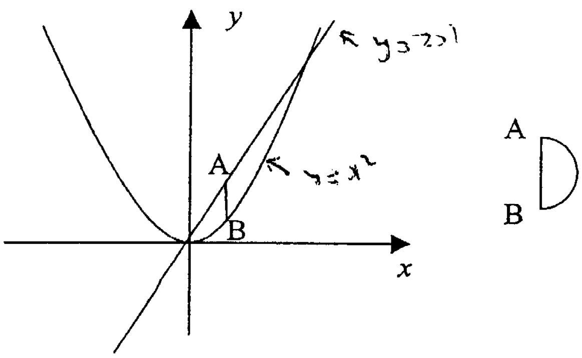
(i) 3 men and 2 women;

(ii) 2 men and 3 women;

(iii) n men and $n + 1$ women?

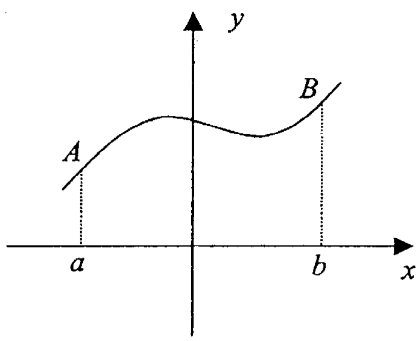
Question 6: START A NEW BOOKLET — Marks

(a) The base of a solid is the region enclosed by $y = 2x$ and $y = x^2$. Cross sections taken perpendicular to the x axis are semicircles with the diameter in the base of the solid (as indicated the diameter AB of the semicircle is perpendicular to the x axis; the semicircle is perpendicular to the xy plane). 5



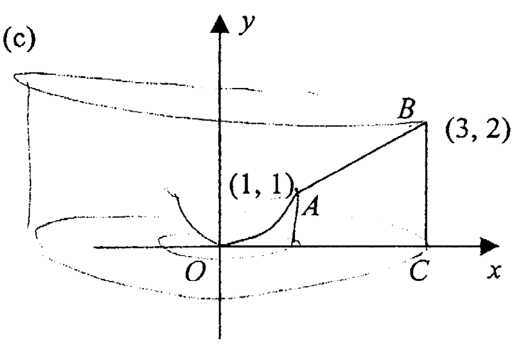
Find the volume of the solid.

(b) The length of the arc AB on the curve $y = f(x)$ between $x = a$ and $x = b$ is given by $l = \int_a^b \sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx$. 4



Find the length of the arc on $y = x^{\frac{3}{2}}$ between $x = 0$ and $x = 4$.

(c) 6

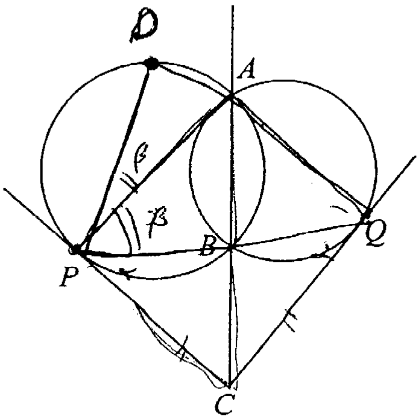


OA is an arc of the parabola $y = x^2$. The region $OABC$ is rotated about the y axis forming a bowl. By using cylindrical shells determine the holding capacity of the bowl.

Question 7: START A NEW BOOKLET Marks

(a) Find the value of a given that $\left(\sqrt{x}+\frac{a}{x}\right)^{10}$ has 13440 as coefficient of x^{-4}. 3

(b) 5



Two circles intersect at A and B. AB is produced to a point C, such that CP and CQ are tangents to the circles as shown and PBQ is a straight line.

NOTE: The diagram is not drawn to scale.

(i) Express CP in terms of CB and CA, and hence prove that $CP = CQ$.

(ii) Show that A, P, C and Q are concyclic.

(iii) Let QA produced meet the larger circle at D. Show that PB bisects $\angle CPD$.

(c) Let $T(m,y)=\frac{{}^mC_0}{y}-\frac{{}^mC_1}{y+1}+\frac{{}^mC_2}{y+2}-\ \ldots\ldots\ +(-1)^m\frac{{}^mC_m}{y+m}$. 7

(i) If it is given that $T(k,x)=\dfrac{k!}{x(x+1)(x+2)\ldots(x+k)}$ for a particular value of k, show that

$$T(k,x)-T(k,x+1)=T(k+1,x)$$

(ii) Hence prove, using Mathematical Induction or otherwise, that for $n \geq 1$

$$T(n,x)=\frac{{}^nC_0}{x}-\frac{{}^nC_1}{x+1}+\frac{{}^nC_2}{x+2}-\ \ldots\ \ +(-1)^n\frac{{}^nC_n}{x+n}=\frac{n!}{x(x+1)(x+2)\ldots(x+n)}$$

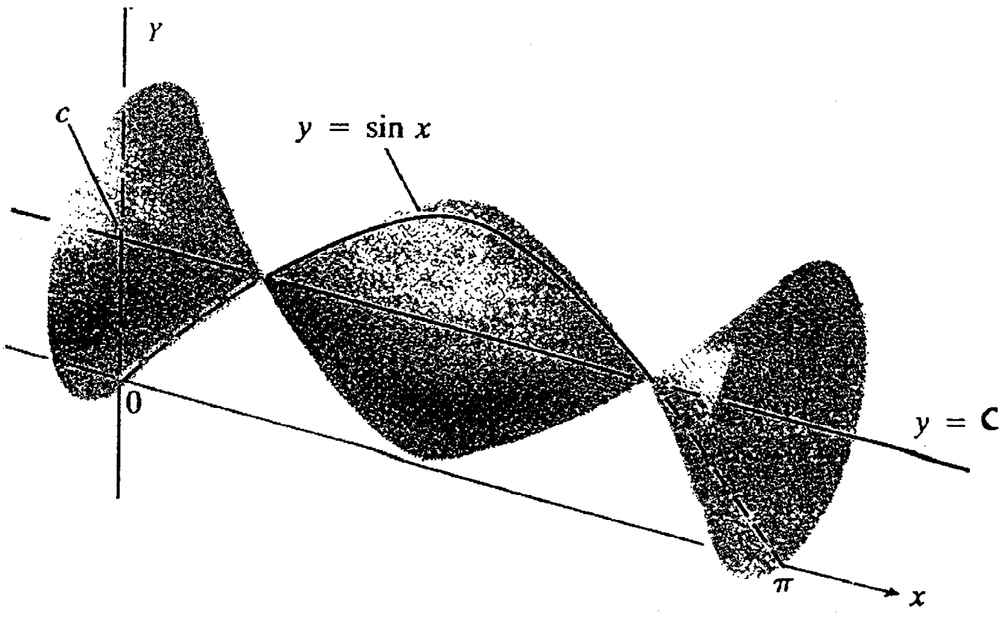
(NOTE: you may use without proof the result ${}^{m+1}C_r={}^mC_r+{}^mC_{r-1}$)

(iii) Hence prove that

$$\frac{{}^nC_0}{1}-\frac{{}^nC_1}{3}+\frac{{}^nC_2}{5}-\ \ldots\ \ +(-1)^n\frac{{}^nC_n}{2n+1}=\frac{2^n n!}{1.3.5\ldots(2n+1)}$$

Question 8: START A NEW BOOKLET Marks

(a) 5



The arch $y = \sin x, \quad 0 \leq x \leq \pi$ is revolved around the line $y = c$ to generate the solid shown. Find the value of c that minimises the volume.

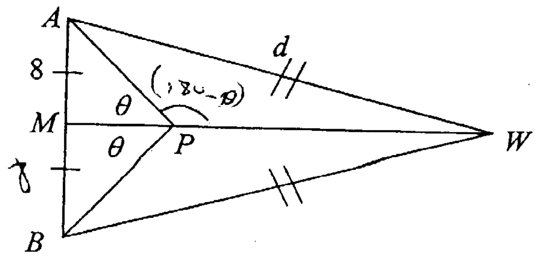
Question 8 is continued on Page 10

(b) (i) Let $f(\theta)=\dfrac{2-\cos\theta}{\sin\theta}$, $0<\theta<\dfrac{\pi}{2}$.

Show that $f'(\theta)=\dfrac{1-2\cos\theta}{\sin^2\theta}$.

Find the minimum value of $f(\theta)$.

(ii) Two towns A and B are 16km apart, and each at a distance of d km from a water well at W. Let M be the midpoint of AB, P be a point on the line segment MW, and $\theta = \angle APM = \angle BPM$. The two towns are to be supplied with water from W, via three straight water pipes: PW, PA and PB as shown below.



Show that the total length of the water pipe L is given by $L=8f(\theta)+\sqrt{d^2-64}$, when $\dfrac{8}{d}\le\sin\theta\le 1$, where $f(\theta)$ is given in part (i).

(iii) If $d=20$, find the length of MP when L is minimum, and the minimum value of L.

Show that this minimum value of L is less than the sum of any pair of sides of ΔABW.

(iv) If $d=9$, show that the minimum value of L cannot be found by using the same methods as used in part (iii). Explain briefly how to find the minimum value of L in this case. (Hint: Draw a diagram which illustrates this situation.)

END OF PAPER