SYDNEY BOYS HIGH
MOORE PARK, SURRY HILLS

2002
HIGHER SCHOOL CERTIFICATE
TRIAL EXAMINATION

Mathematics Extension 2

General Instructions
- Reading time — 5 minutes
- Working time — 3 hours
- Write using black or blue pen
- Board-approved calculators may be used
- A table of standard integrals is provided at the back of this paper
- All necessary working should be shown in every question

Total marks — 120
- Attempt questions 1—8
- All questions are of equal value, the mark value is shown beside each part.

Examiner: D.M.Hespe

Note: This is an assessment task only and does not necessarily reflect the content or format of the Higher School Certificate.

Total marks – 120
Attempt Questions 1–8
All questions are of equal value

Answer each question in a SEPARATE writing booklet. Extra writing booklets are available.

Marks

Question 1 (15 marks) Use a SEPARATE writing booklet.

(a) Find

(i) $\int \sin^{-1} x \, dx$ **2**

(ii) $\int \frac{x}{1+x^4} \, dx$ **2**

(iii) $\int \tan^3 x \, dx$ **2**

(b) Evaluate $\int_0^{\frac{\pi}{2}} \frac{dx}{1+\cos x}$ using the substitution $t = \tan\frac{x}{2}$. **3**

(c) Given that $I_n = \int_1^e (\ln x)^n \, dx$, $n = 0, 1, 2, \ldots$, **3**

show that $I_n = e - nI_{n-1}$.

(d) If $x = \frac{\pi}{4} - u$,

(i) Show that $\tan x = \frac{1-\tan u}{1+\tan u}$. **1**

(ii) Hence or otherwise, show that $\int_0^{\frac{\pi}{4}} \ln(1+\tan x) \, dx = \frac{\pi}{8} \ln 2$. **2**

Marks

Question 2 (15 marks) Use a SEPARATE writing booklet.

(a) Explain the flaw in this "proof" that $i = -i$. **2**

$$i = i$$
$$\sqrt{-1} = \sqrt{-1}$$
$$\sqrt{\frac{-1}{1}} = \sqrt{\frac{1}{-1}}$$
$$\frac{\sqrt{-1}}{\sqrt{1}} = \frac{\sqrt{1}}{\sqrt{-1}}$$
$$\frac{i}{1} = \frac{1}{i} = \frac{-(-1)}{i} = \frac{-i^2}{i} = -i$$
$$\therefore\ i = -i$$

(b) $u = -3 - 4i$ and $v = 1 - i$ are two complex numbers. Express in the form $x + iy$, where x and y are real:

(i) $\bar{u} - v$ **1**

(ii) $\dfrac{2u}{v}$ **2**

(iii) $\sqrt{u}$ **2**

(c) On an Argand diagram sketch the region defined by **2**

$$\{ z : -\tfrac{\pi}{6} \le \arg z \le \tfrac{\pi}{6} \cap |z| \le 1 \}.$$

(d) (i) If a, b are the complex numbers represented by the points A and B on an Argand diagram, what geometrical properties correspond to the modulus and argument of $\dfrac{b}{a}$? **2**

(ii) Show that, if the four points representing the complex numbers z_1, z_2, z_3, and z_4 are concyclic, the fraction $\dfrac{(z_1 - z_2)(z_3 - z_4)}{(z_3 - z_2)(z_1 - z_4)}$ must be real. **4**

Marks

Question 3 (15 marks) Use a SEPARATE writing booklet.

(a) Reduce to irreducible factors over the complex field: $x^3 - 4x^2 + 7x - 6$. **3**

(b) Find the polynomial $f(x)$ of the fourth degree such that $f(0) = f(1) = 1$, $f(2) = 13$, $f(3) = 73$ and $f'(0) = 0$. **4**

(c) (i) Prove that if $P(x)$ has a root of multiplicity m, then $P'(x)$ has a root of multiplicity $m-1$. **2**

(ii) Find the value of c if the polynomial $5x^5 - 3x^3 + c$ has a positive repeated root. **3**

(d) Let α, β, γ be the roots of the equation $x^3 + px + q = 0$, where $q \neq 0$. Find, in terms of p and q, the coefficients of the cubic equation whose roots are α^{-1}, β^{-1}, and γ^{-1}. **3**

Marks

Question 4 (15 marks) Use a SEPARATE writing booklet.

(a) Solve the simultaneous equations: **4**

$$x^2 + xy + y^2 = 7,$$

$$2x^2 - xy + y^2 = 28.$$

(b) Show that if $b^2 < 4ac$, the value of the function $ax^2 + bx + c$ will have the same sign as a for all real values of x. **2**

(c) (i) By considering the expression $x^2 - 2xy + 5y^2 + 2x - 14y + k$ as a quadratic function of x, show that it is positive for all real values of x and y if $k > 10$. **4**

(ii) Show that if $k = 10$, the expression may be written in the form $(x + py + q)^2 + (ry + s)^2$, and hence find the simultaneous values of x and y for which the expression is zero. **4**

(iii) Deduce the minimum value of the expression for a general value of k. **1**

Marks

Question 5 (15 marks) Use a SEPARATE writing booklet.

(a) A particle P of mass m starts from rest at a point O and falls under gravity in a medium where the resistance to its motion has magnitude mkv, v being the speed of the particle and k is a constant.

(i) Draw a diagram to show the *forces* acting on the particle during this downward path, and hence write down the equation of motion. **2**

(ii) Show that the expression for its velocity v at any time t is given by **2**

$$v = \frac{g}{k}\left(1 - e^{-kt}\right).$$

(iii) Explain what is meant by the *terminal velocity* and find an expression for the terminal velocity V_T. **3**

(b) A second particle Q, also of mass m, is fired vertically upwards from O with initial speed u, so that P and Q leave O simultaneously.

(i) Draw a diagram to show the *forces* acting on the particle during this downward path, and hence write down the equation of motion. **2**

(ii) Find an expression for the time t when Q comes to rest. **3**

(c) Show that, at the instant Q comes to rest, the velocity of P is given by: **3**

$$v = \frac{V_T u}{V_T + u}.$$

Marks

Question 6 (15 marks) Use a SEPARATE writing booklet.

(a) (i) Show that if x is real, the expression $\frac{(x-2)^2}{x-1}$ cannot take any value between -4 and 0. **2**

(ii) Sketch the graph of the expression. **3**

(iii) Show that the equation $\frac{(x-2)^2}{x-1} = \frac{k}{x}$ has three real roots if k is positive, but only one real root if k is negative. **2**

(b) For $z = r(\cos\theta + i\sin\theta)$, find r and the smallest positive value of θ which satisfy the equation $2z^3 = 9 + 3\sqrt{3}i$. **2**

(c) Using the method of shells, or otherwise, find the volume of the solid formed when the region bounded by the curve $y = x^2 + 1$ and the x-axis between $x = 0$ and $x = 2$ is rotated about the y-axis. **3**

(d) Explain why, if $\lim_{n\to\infty}\left(\sqrt{n^2+n} - n\right) = \lim_{n\to\infty}\left(n \times \sqrt{1+\frac{1}{n}} - n\right)$, then the limit is not zero, but a half. **3**

Marks

Question 7 (15 marks) Use a SEPARATE writing booklet.

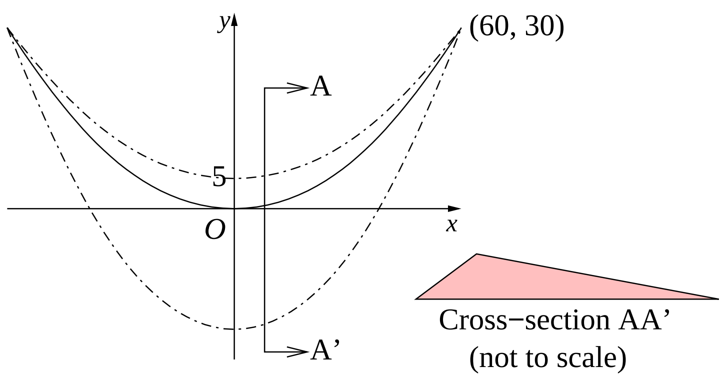
(a) Find the rational roots of $x^4+2x^3-17x^2-18x+32=0$ using the substitution $y=x^2+x$, or otherwise. **2**

(b) (i) Prove that the medians of a triangle are concurrent at a point which is a point of trisection of each median. [A *median* of a triangle is a line from a vertex to the mid point of the opposite side.] **3**

(ii) If the medians of triangle ABC meet at G, and AG is produced to K so that $AG=GK$, prove that the triangle BGK is similar to the triangle whose sides are equal in length to the three medians. **3**

(iii) Also show that the area of the triangle whose sides are equal in length to the medians is ¾ of the area of triangle ABC. **2**

(c) **5**



Barcan sand dunes are parabolic in plan view and are triangular in cross section with the inner face having an angle of repose of 35° to the horizontal and the outer face at 10° to the horizontal. The figure above shows one such dune (dimensions are in metres). Calculate the volume of sand.

Marks

Question 8 (15 marks) Use a SEPARATE writing booklet.

(a) (i) Show that the length, ℓ, of a curve, $y = f(x)$, is given by **2**

$$\ell = \int \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx.$$

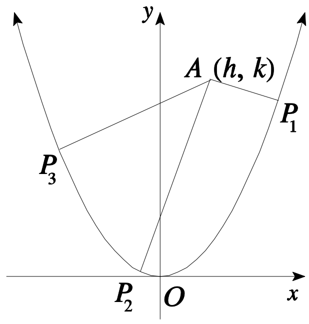
(ii) A circular disc, centre A, of radius a, rolls without slipping along the axis of x. Initially the point P on the edge of the disc is at the origin of coordinates. Prove that, when the radius AP has turned through an angle θ, the coordinates of P are: **3**
$x = a(\theta - \sin\theta),\ y = a(1 - \cos\theta)$.

(iii) When P is again in contact with the axis of x, prove that the length of its path is $8a$. **2**

(b) Sum the series, n being a positive integer, **3**

$$1 + x\cos x + x^2\cos 2x + x^3\cos 3x + \cdots + x^n\cos nx.$$

(c) (i)



Prove that, in general, three normals can be drawn from any point to a parabola. **3**

(ii) Also show that if P_1, P_2, and P_3 have coordinates (x_1, y_1), (x_2, y_2), and (x_3, y_3) respectively, then $x_1 + x_2 + x_3 = 0$. **2**

End of paper

STANDARD INTEGRALS

$$\int x^n\,dx = \frac{1}{n+1}x^{n+1}, \quad n \neq -1; \quad x \neq 0, \text{ if } n<0$$

$$\int \frac{1}{x}\,dx = \ln x, \quad x>0$$

$$\int e^{ax}\,dx = \frac{1}{a}e^{ax}, \quad a \neq 0$$

$$\int \cos ax\,dx = \frac{1}{a}\sin ax, \quad a \neq 0$$

$$\int \sin ax\,dx = -\frac{1}{a}\cos ax, \quad a \neq 0$$

$$\int \sec^2 ax\,dx = \frac{1}{a}\tan ax, \quad a \neq 0$$

$$\int \sec ax \tan ax\,dx = \frac{1}{a}\sec ax, \quad a \neq 0$$

$$\int \frac{1}{a^2+x^2}\,dx = \frac{1}{a}\tan^{-1}\frac{x}{a}, \quad a \neq 0$$

$$\int \frac{1}{\sqrt{a^2-x^2}}\,dx = \sin^{-1}\frac{x}{a}, \quad a1>0, \quad -a<x<a$$

$$\int \frac{1}{\sqrt{x^2-a^2}}\,dx = \ln\left(x+\sqrt{x^2-a^2}\right), \quad x>a>0$$

$$\int \frac{1}{\sqrt{x^2+a^2}}\,dx = \ln\left(x+\sqrt{x^2+a^2}\right)$$

NOTE : $\ln x = \log_e x, \quad x>0$