SYDNEY BOYS HIGH SCHOOL
MOORE PARK, SURRY HILLS

2004

TRIAL HIGHER SCHOOL CERTIFICATE EXAMINATION

Mathematics Extension 2

General Instructions

- Reading time – 5 minutes.
- Working time – 3 hours.
- Write using black or blue pen.
- Board approved calculators may be used.
- All *necessary* working should be shown in every question if full marks are to be awarded.
- Marks may **NOT** be awarded for messy or badly arranged work.
- Hand in your answer booklets in **3** sections.
  Section A (Questions 1 - 3),
  Section B (Questions 4 - 5) and
  Section C (Questions 6 - 8).
- Start each section in a **NEW** answer booklet.

Total Marks - 120 Marks

- Attempt Sections A - C
- All questions are NOT of equal value.

Examiner: *E. Choy*

This is an assessment task only and does not necessarily reflect the content or format of the Higher School Certificate.

Total marks – 120
Attempt Questions 1 – 8
All questions are of equal value

Answer each section in a SEPARATE writing booklet. Extra writing booklets are available.

SECTION A (Use a SEPARATE writing booklet)

Question 1 (15 marks) **Marks**

(a) Evaluate

(i) $\displaystyle\int_0^{\sqrt{3}} \frac{dx}{\sqrt{4-x^2}}$ **1**

(ii) $\displaystyle\int_0^{1} \sqrt{4-x^2}\, dx$ **1**

(iii) $\displaystyle\int_{-1}^{2} x\sqrt{2-x}\, dx$ **1**

(b) Evaluate

(i) $\displaystyle\int_1^{2} \frac{e^{2x}}{e^x-1} dx$ **2**

(ii) $\displaystyle\int_0^{\frac{\pi}{2}} \frac{1}{4+5\sin x} dx$ **4**

(c) (i) If $I_n = \displaystyle\int_0^{\frac{\pi}{4}} \tan^{2n} x\, dx$, $n \geq 0$, show that $I_n + I_{n-1} = \dfrac{1}{2n-1}$ **3**

(ii) Hence, evaluate $\displaystyle\int_0^{\frac{\pi}{4}} \tan^6 x\, dx$ **1**

(d) Evaluate $\displaystyle\int_1^{e} x\ln\left(x^2\right) dx$ **2**

Question 2 (15 marks) **Marks**

(a) (i) Sketch on the same axes the graphs **2**

$$y = x + 3 \text{ and } y = 2|x|.$$

(ii) Hence or otherwise:

(α) Solve for x, $2|x| < x + 3$. **2**

(β) Sketch the curve $y = \dfrac{2|x|}{x+3}$. **3**

(b) Let $f(x) = \dfrac{3}{x-1}$.

On separate diagrams sketch the graphs of the following:

(i) $y = f(|x|)$ **2**

(ii) $y^2 = f(x)$ **3**

(iii) $y = e^{f(x)}$ **3**

SECTION A continued

Question 3 (15 marks) **Marks**

(a) If $z=-1+i\sqrt{3}$ and $w=2\text{cis}\dfrac{\pi}{6}$

(i) Find $|z|$. **1**

(ii) $\arg z$. **1**

(iii) Express z in the form $r\text{cis}\theta$. **1**

(iv) Express $z^6 \div w^3$ in the form $r\text{cis}\theta$. **1**

(b) (i) Express $\sqrt{5-12i}$ in the form $a+ib$. **2**

(ii) Hence describe the locus of the point which represents z on the Argand diagram if **2**

$$\left|z^2-5+12i\right|=\left|z-3+2i\right|$$

(c) The origin and the points representing the complex numbers z, $\dfrac{1}{z}$ and $z+\dfrac{1}{z}$ are joined to form a quadrilateral.
Write down the conditions for z so that the quadrilateral will be

(i) a rhombus; **1**

(ii) a square. **1**

(d) (i) Find the equation and sketch the locus of z if **2**

$$\left|z-i\right|=\text{Im}\left(z\right)$$

(ii) Find the least value of $\arg z$ in (i) above. **3**

END OF SECTION A

SECTION B (Use a SEPARATE writing booklet)

Question 4 (15 marks) **Marks**

(a) $3-i$ is a zero of $P(z)=z^3-4z^2-2z+m$, where m is a real number. **3**

Find m.

(b) If α, β and γ are the roots of $x^3+px+q=0$, find a cubic equation whose roots are α^2, β^2 and γ^2. **3**

(c) Given a real polynomial $Q(x)$, show that if α is a root of $Q(x)-x=0$, then α is also a root of $Q(Q(x))-x=0$. **3**

(d) Use the following identity to answer the following questions.

$$\cos 5\theta = 16\cos^5\theta - 20\cos^3\theta + 5\cos\theta$$

(i) Solve $16x^5-20x^3+5x=0$ **3**

(ii) Hence show that **3**

$$\cos\frac{\pi}{10}\cos\frac{3\pi}{10}\cos\frac{7\pi}{10}\cos\frac{9\pi}{10}=\frac{5}{16}$$

Question 5 (15 marks) **Marks**

(a) Let $z=\cos\theta+i\sin\theta$, show that

(i) $z^n+z^{-n}=2\cos n\theta$ **1**

(ii) $z^n-z^{-n}=2i\sin n\theta$ **1**

(b) (i) Show that for any integer k that **2**

$$\left[z-\left(\cos\frac{k\pi}{4}+i\sin\frac{k\pi}{4}\right)\right]\left[z-\left(\cos\frac{(8-k)\pi}{4}+i\sin\frac{(8-k)\pi}{4}\right)\right]=z^2.$$

(ii) Hence simplify the following products

(α) $\left[z-\left(\cos\frac{\pi}{4}+i\sin\frac{\pi}{4}\right)\right]\left[z-\left(\cos\frac{7\pi}{4}+i\sin\frac{7\pi}{4}\right)\right]$ **1**

(β) $\left[z-\left(\cos\frac{3\pi}{4}+i\sin\frac{3\pi}{4}\right)\right]\left[z-\left(\cos\frac{5\pi}{4}+i\sin\frac{5\pi}{4}\right)\right]$ **1**

(c) Using the results of (b) above, factorise z^4+1 into 2 real quadratic factors. **2**

(d) Using (a) and (c) above, prove the identity **2**

$$\cos 2\theta=2\cos^2\theta-1$$

(e) The complex numbers $z=x+iy$, $z_1=-x+iy$ and $z_2=-\frac{2}{z}$ are represented by the points P, P_1 and P_2 in the Argand diagram respectively.

(i) Show that O, P_1 and P_2 are collinear where O is the origin. **3**

(ii) Show that $OP_1\times OP_2=2$ **2**

END OF SECTION B

SECTION C (Use a SEPARATE writing booklet)

Question 6 (15 marks) **Marks**

(a) A particle of mass *m* is projected vertically upwards with a velocity of $u\ \text{ms}^{-1}$, with air resistance proportional to its velocity.

(i) Show that after a time *t* seconds, the height above the ground is **4**

$$x_1 = \frac{g+ku}{k^2}\left(1-e^{-kt}\right)-\frac{gt}{k},$$

where *k* is a constant and *g* is the acceleration due to gravity.

(ii) At the same time another particle of mass *m* is released from rest, from a height *h* metres vertically above the first particle. You may assume that at time *t* seconds, its distance from the ground is given by: **4**

$$x_2 = h+\frac{g}{k^2}\left(1-e^{-kt}\right)-\frac{gt}{k}$$

Show that the two particles will meet at time *T* where

$$T=\frac{1}{k}\ln\left(\frac{u}{u-kh}\right)$$

(b) A vehicle of mass *m* moves in a straight line subject to a resistance $P+Qv^2$, where *v* is the speed and *P* and *Q* are constants with $Q>0$.

(i) Form an equation of motion for the acceleration of the vehicle. **1**

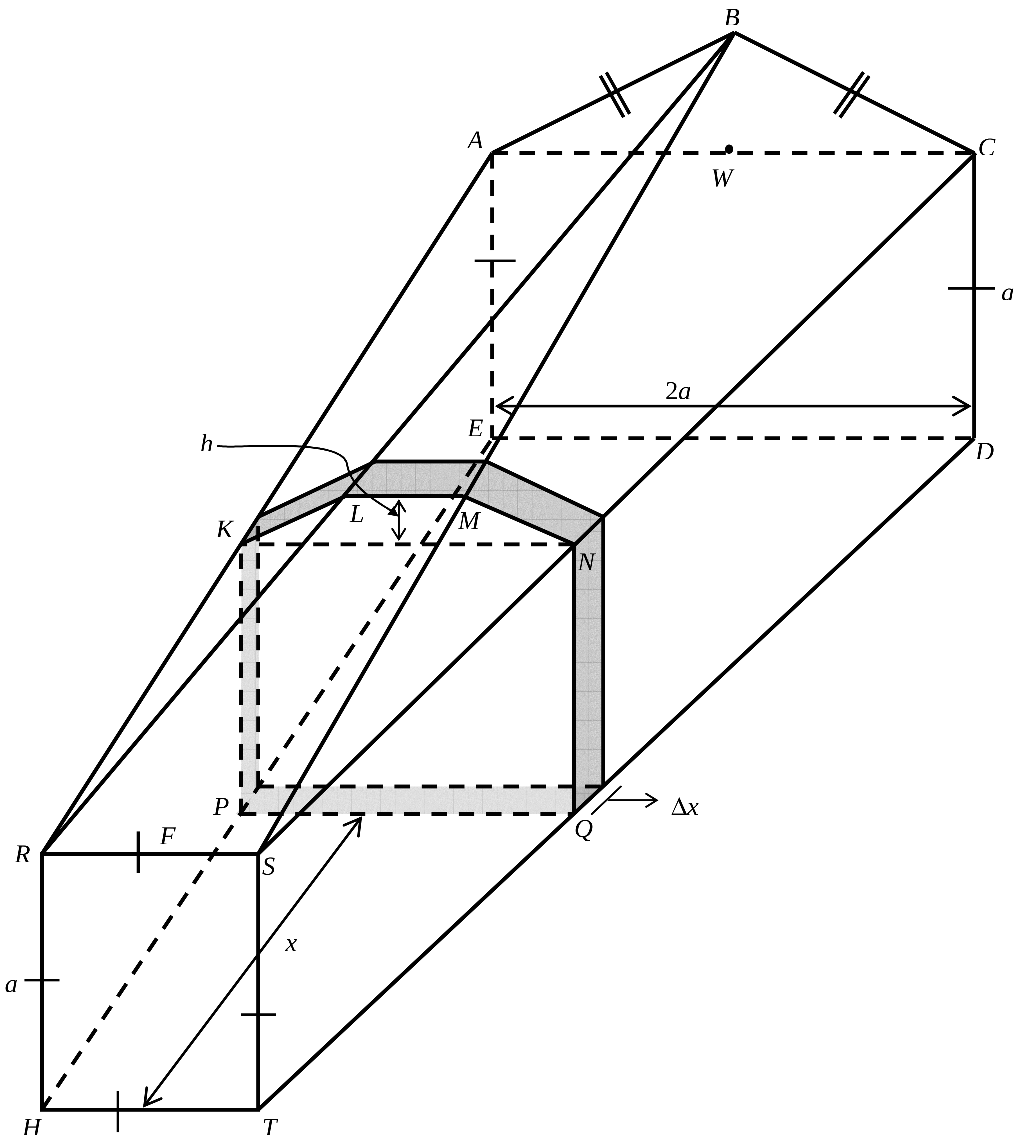
(ii) Hence show that if $P=0$, the distance required to slow down from speed $\frac{3U}{2}$ to speed *U* is $\frac{m}{Q}\ln\left(\frac{3}{2}\right)$. **3**

(iii) Also show that if $P>0$, the distance required to stop from speed *U* is given by **3**

$$D=\lambda\ln\left(1+kU^2\right)$$

where *k* and λ are constants

Question 7 (15 marks) **Marks**



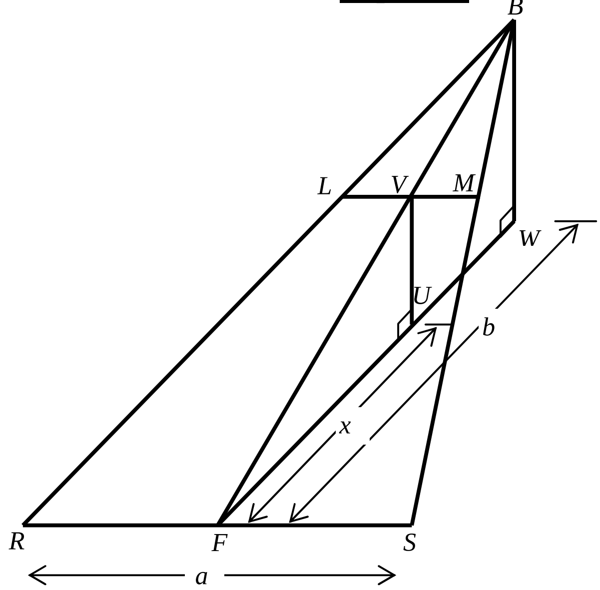
The diagram above shows a solid with a trapezoidal base *EDTH* of length *b* metres.
The front end *HTSR* is a square with side length *a* metres.
The back is the pentagon *ABCDE* which consists of the rectangle *ACDE* with length 2*a* metres and width *a* metres, surmounted by the equilateral triangle *ABC*.

Consider a slice of the solid, parallel to the front and the back, with face formed by both the trapezium *KLMN* and the rectangle *KNQP*, which has thickness Δx and is at a distance *x* metres from *HT*.

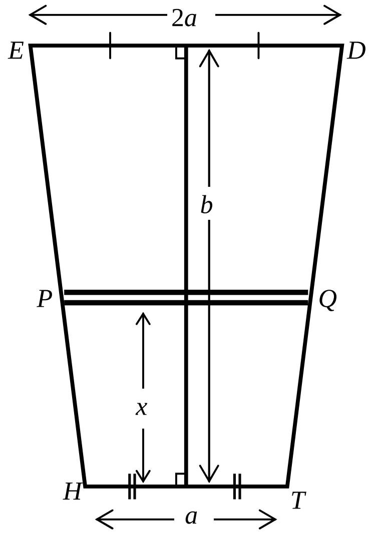
Question 7 continued on page 9

(i) Show that the height, *BW*, of the equilateral triangle *ABC* is $\sqrt{3}a$ metres. **2**

Top View



Base of Solid



(ii) Given that the perpendicular height of the trapezium *KLMN* is *h* metres ie *VU* = *h*, use the similar triangles *BWF* and *VUF*, in the Top View, to find *h* in terms of *a*, *b* and *x*. **3**

(iii) Given that the triangles *BLM* and *BRS* are similar, show that **3**

$$LM = \frac{a(b-x)}{b}$$

(iv) Using the cross section of the base, find the length of *PQ* in terms of *a*, *b* and *x*. **3**

(v) Find the volume of the solid. **4**

Question 8 starts on page 10

SECTION C continued

Question 8 (15 marks) **Marks**

(a) If $a > 0$, $b > 0$ and $a + b = t$ show that **3**

$$\frac{1}{a} + \frac{1}{b} \geq \frac{4}{t}$$

(b) There are n ($n > 1$) different boxes each of which can hold up to $n + 2$ books. Find the probability that:

(i) No box is empty when n different books are put into the boxes at random. **1**

(ii) Exactly one box is empty when n different books are put into the boxes at random. **2**

(iii) No box is empty when $n + 1$ different books are put into the boxes at random. **2**

(iv) No box is empty when $n + 2$ different books are put into the boxes at random. **2**

(c) *PQRS* is a cyclic quadrilateral such that the sides *PQ, QR, RS* and *SP* touch a circle at *A*, *B, C* and *D* respectively.

Prove that:

(i) *AC* is perpendicular to *BD.* **2**

(ii) Let the midpoints of *AB*, *BC, CD* and *DA* be *E*, *F*, *G* and *H* respectively. **3**
Show that *E, F, G* and *H* lie on a circle.

End of paper

STANDARD INTEGRALS

$$\int x^n dx = \frac{1}{n+1}x^{n+1},\ n \neq -1;\ x \neq 0, \text{if } n < 0$$

$$\int \frac{1}{x} dx = \ln x,\ x > 0$$

$$\int e^{ax} dx = \frac{1}{a}e^{ax},\ a \neq 0$$

$$\int \cos ax\, dx = \frac{1}{a}\sin ax,\ \ a \neq 0$$

$$\int \sin ax\, dx = -\frac{1}{a}\cos ax,\ \ a \neq 0$$

$$\int \sec^2 ax\, dx = \frac{1}{a}\tan ax,$$

$$\int \sec ax \tan ax\, dx = \frac{1}{a}\sec ax,\ \ a \neq 0$$

$$\int \frac{1}{a^2 + x^2} dx = \frac{1}{a}\tan^{-1}\frac{x}{a},\ \ a \neq 0$$

$$\int \frac{1}{\sqrt{a^2 - x^2}} dx = \sin^{-1}\frac{x}{a},\ \ a > 0,\ -a < x < a$$

$$\int \frac{1}{\sqrt{x^2 - a^2}} dx = \ln\left(x + \sqrt{x^2 - a^2}\right),\ x > a > 0$$

$$\int \frac{1}{\sqrt{x^2 + a^2}} dx = \ln\left(x + \sqrt{x^2 + a^2}\right)$$

NOTE: $\ln x = \log_e x,\ \ x > 0$