



SYDNEY BOYS HIGH SCHOOL
MOORE PARK, SURRY HILLS

2006
TRIAL HIGHER SCHOOL
CERTIFICATE

Mathematics Extension 2

General Instructions

- Reading time – 5 minutes.
- Working time – 3 hours.
- Write using black or blue pen.
- Board approved calculators may be used.
- All *necessary* working should be shown in every question if full marks are to be awarded.
- Marks may **NOT** be awarded for messy or badly arranged work.
- Hand in your answer booklets in 3 sections.
Section A (Questions 1 - 2),
Section B (Questions 3 - 4)
Section C (Questions 5 - 6)
Section D (Questions 7 - 8).
- Start each **NEW** section in a separate answer booklet.

Total Marks - 120 Marks

- Attempt Sections A - D
- All questions are of equal value.

Examiner: *E. Choy*

This is an assessment task only and does not necessarily reflect the content or format of the Higher School Certificate.

Total marks – 120
Attempt Questions 1 - 8
All questions are of equal value

Answer each section in a SEPARATE writing booklet. Extra writing booklets are available.

SECTION A (Use a SEPARATE writing booklet)

Question 1 (15 marks)	Marks
(a) By first completing the square, evaluate the following integrals	
(i) $\int_{-1}^0 \frac{dx}{\sqrt{3-2x-x^2}}$	2
(ii) $\int_0^1 \sqrt{x(1-x)} dx$	2
(b) Integrate the expressions below	
(i) $\int \frac{1}{x \ln x} dx$	1
(ii) $\int x \ln x dx$	2
(iii) $\int \frac{x+1}{x^2+x+1} dx$	2
(c) Use the technique of <i>integration by parts</i> to evaluate	2
$\int_0^{\frac{1}{2}} \cos^{-1} x dx$	
(d) (i) Find real numbers A , B , and C so that	2
$\frac{10}{(3+x)(1+x^2)} = \frac{A}{3+x} + \frac{Bx+C}{1+x^2}$	
for all $x \neq -3$	
(ii) Use part (i) above and the substitution $t = \tan \theta$ to find	2
$\int \frac{10d\theta}{3 + \tan \theta}$	

SECTION A continued

Question 2 (15 marks)

Marks

- | | | | |
|-----|-------|--|---|
| (a) | (i) | Write the complex number $-\sqrt{3} + i$ in modulus-argument form. | 1 |
| | (ii) | Hence, use de Moivre's Theorem to find $(-\sqrt{3} + i)^{10}$ in the form $a + ib$, for real values a and b . | 2 |
| | | Sketch each of the following regions on separate Argand diagrams | |
| | (i) | $-1 < \operatorname{Re}(z) < 2$ and $0 < \operatorname{Im}(z) < 3$ | 2 |
| | (ii) | $z\bar{z} - (1-i)z - (1+i)\bar{z} < 2$ | 2 |
| | (iii) | $0 < \arg[(1-i)z] < \frac{\pi}{6}$ | 2 |
| | | Find the square roots of the complex number $-3 + 4i$ | 2 |
| | (ii) | Find the roots of the quadratic equation $x^2 - (4 - 2i)x + (6 - 8i) = 0$ | 2 |
| | | The locus of a point P , which moves in the complex plane, is represented by the equation $ z - (3 + 4i) = 5$ | |
| | (i) | Sketch the locus of the point P . | 2 |
| | (ii) | Find the modulus of z when $\arg z = \tan^{-1}(\frac{1}{2})$. | 2 |

SECTION B (Use a SEPARATE writing booklet)

Question 3 (15 marks)

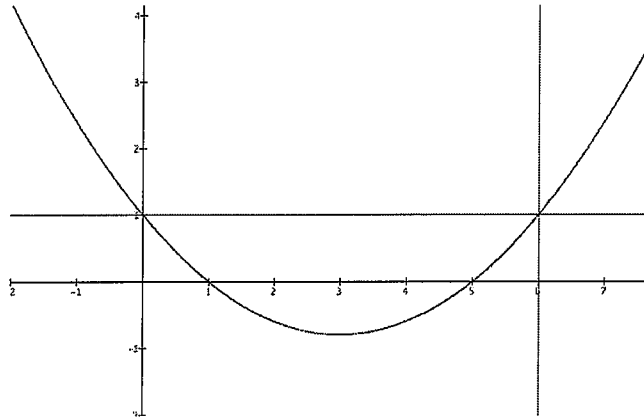
- | | | | |
|-----|---|---|---|
| (a) | Find a cubic equation with roots α , β and γ such that | 3 | |
| | $\left. \begin{array}{l} \alpha\beta\gamma = 5 \\ \alpha + \beta + \gamma = 7 \\ \alpha^2 + \beta^2 + \gamma^2 = 29 \end{array} \right\}$ | | |
| | The polynomial $P(x)$ is defined by $P(x) = x^4 + Ax^2 + B$, where A and B are real positive numbers. | | |
| | (i) | Explain why $P(x)$ has no real zeroes. | 2 |
| | (ii) | If two of the zeroes of $P(x)$ are ib and $-id$ where b and d are real show that: | 4 |
| | | $b^4 + d^4 = A^2 - 2B$ | |
| | | Given that $f(x) = x^3 - 3ax + b$, where a and b are real numbers then: | |
| | (i) | Show that $y = f(x)$ has turning points if $a > 0$, and find their coordinates. | 3 |
| | (ii) | Show that $f(x)$ has three distinct real zeroes if $b^2 < 4a^3$. | 3 |

SECTION B continued

Question 4 (15 marks)

Marks

(a)



The sketch above shows the parabola $y = f(x)$, where

$$f(x) = \frac{1}{5}(x-1)(x-5).$$

Without any use of calculus, draw careful sketches of the following curves, showing all intercepts, asymptotes and turning points.

NB The vertex of the parabola is at $\left(3, -\frac{4}{5}\right)$.

- | | | |
|-------|-----------------------|---|
| (i) | $y = \frac{1}{f(x)}$ | 2 |
| (ii) | $y = [f(x)]^2$ | 3 |
| (iii) | $y = \tan^{-1}[f(x)]$ | 3 |
| (iv) | $y = f(\ln x)$ | 3 |

(b) Suppose the function $f(x) = O(x) + E(x)$, where $O(x)$ is odd and $E(x)$ is even.

- | | | |
|------|--|---|
| (i) | By considering $f(-x)$, find an expression for $O(x)$ in terms of f . | 2 |
| (ii) | Hence write down $O(x)$ when $f(x) = e^x$. | 2 |

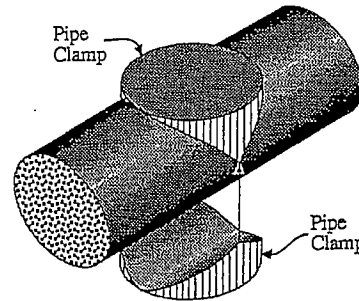
SECTION C starts on page 5

SECTION C (Use a SEPARATE writing booklet)

Question 5 (15 marks)

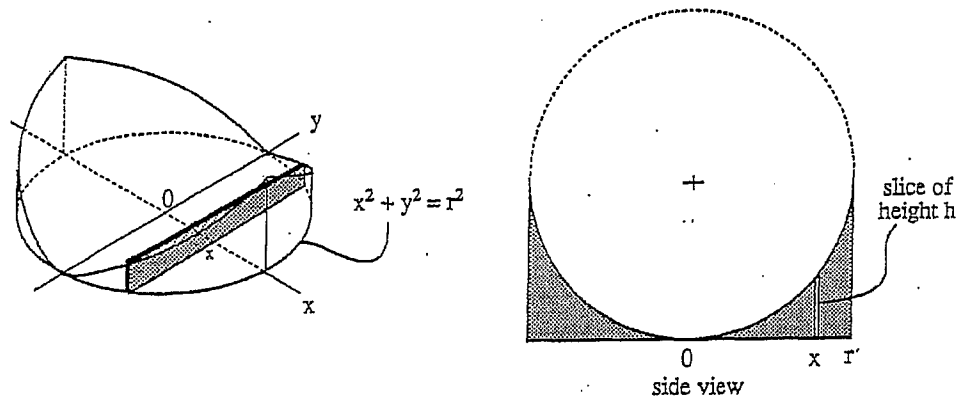
Marks

- (a) A pipe-clamp is made of two identical pieces. Each piece has a circular base of radius r units and the other face is curved so as to fit flush against the pipe held between the two pieces.

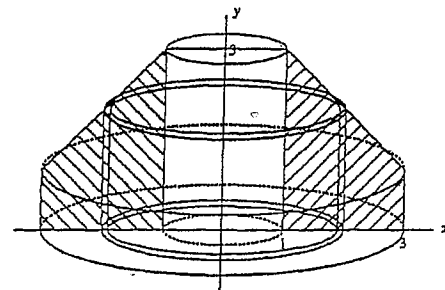


The pipe also has a radius of r units.

A vertical slice, of thickness Δx , taken x units from the centre of the base is in the shape of a rectangle with one side in the circular base and of height necessary to reach the cylindrical pipe as shown in the diagram below:



- (i) Show that the height of the slice taken x units from O is given by 3
- $$h = r - \sqrt{r^2 - x^2}$$
- (ii) Show that the volume, ΔV , of such a slice is given by 3
- $$\Delta V \approx \left[2r\sqrt{r^2 - x^2} - 2(r^2 - x^2) \right] \Delta x$$
- (iii) Hence, find by integration, the volume of ONE piece of the pipe-clamp. 3
- (b) (i) Show that the volume, ΔV , of a right cylindrical shell of height H , with inner radius r and thickness Δr is given by the formula 2
- $$\Delta V = 2\pi r H \Delta r$$
- where Δr is sufficiently small so that $(\Delta r)^2$ may be neglected.
- (b) (ii) A metal umbrella base is formed by rotating the area enclosed between $x = 1$, $x = 3$, $y = 0$ and $y = 4 - x$ about the y -axis as shown. 4



Using the method of cylindrical shells, find the volume of the umbrella base.

SECTION C continued

Question 6 (15 marks)

- (a) A point T moves so that the sum of its distances from the point $(-2, 0)$ and $(2, 0)$, on a Cartesian plane, is 6 units.
- (i) Show that the locus of T is an ellipse \mathcal{E} with the equation 2
- $$\frac{x^2}{9} + \frac{y^2}{5} = 1$$
- (ii) Find the equation of the auxiliary circle, \mathcal{A} , of \mathcal{E} . 1
- (iii) Find the eccentricity, coordinates of the foci and the equations of the directrices of the ellipse, \mathcal{E} . 2
- (iv) Draw a neat sketch, showing the ellipse and its auxiliary circle. 1
- (v) A line parallel to the y -axis meets the positive x -axis at N and the curves \mathcal{E} and \mathcal{A} at P and Q respectively. 1
 Given the coordinates $N(3 \cos \theta, 0)$, find the coordinates of P and Q (where P and Q are in the first quadrant).
- (vi) Find the equations of the tangents at P and Q . 2
- (vii) If R is the point of intersection of the tangents at P and Q : 2
- (α) Show that R lies on the major axis of \mathcal{E} . 2
- (β) Prove that the product of the lengths ON and OR is independent of the positions of P and Q on the curves. 2
- (b) Given p red balls and m yellow balls, where $p - m + 1 > 0$, arranged in a row. 2
 Show that the number of ways of arranging them so that no two yellow balls appear together is given by:

$${}^{p+1}C_m$$

SECTION D starts on page 7

SECTION D (Use a SEPARATE writing booklet)

Question 7 (15 marks)

Marks

- | | | | |
|-----|-------|--|---|
| (a) | (i) | Show that $z^5 + 1 = (z+1)(z^4 - z^3 + z^2 - z + 1)$ | 1 |
| | (ii) | If z is a solution to $z^5 + 1 = 0$ where $z \neq -1$, prove that $1 + z^2 + z^4 = z + z^3$. | 1 |
| | (iii) | Hence show that $\cos \frac{\pi}{5} + \cos \frac{3\pi}{5} = \frac{1}{2}$ | 3 |

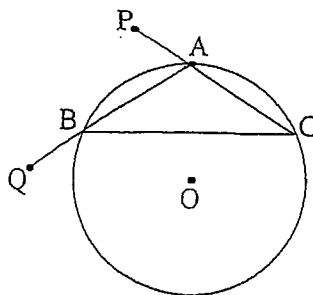
- (b) For integer values of k where $k = 0, 1, 2, \dots$ define I_k as follows:

$$I_k = \int_0^{\frac{\pi}{2}} \cos^k x \, dx$$

- | | | |
|------|--|---|
| (i) | Express I_{k+2} in terms of k and I_k . | 2 |
| (ii) | Hence find an expression for I_{2n} , where $n = 0, 1, 2, \dots$ | 2 |

- (c) In $\triangle ABC$, in the diagram on the right, $AB = AC$.

Produce CA to P and AB to Q so that $AP = BQ$.



- | | | |
|------|--|---|
| (i) | Show that $\angle OAP = \angle OBQ$. | 3 |
| (ii) | Prove that A, P, Q and O , the centre of circle ABC , are concyclic. | 3 |

Question 8 (15 marks)

- (a) A particle is projected **vertically** upwards in a resisting medium where the resistance varies as the square of the velocity and k is the constant of variation. If the velocity of projection is $v_0 \tan \alpha$,

- (i) Show that the maximum height, H , reached is given by: 3

$$H = \frac{1}{2k} \ln \left(\frac{g + kv_0^2 \tan^2 \alpha}{g} \right)$$

- (ii) Show that the particle returns to the point of projection with velocity $v_0 \sin \alpha$ given that v_0 is the terminal velocity. 4

- (iii) Show that the time of ascent is $\frac{v_0 \alpha}{g}$ 3

- (iv) Show that the time of descent is $\frac{v_0}{g} \ln(\sec \alpha + \tan \alpha)$ 2

- (b) Prove by induction that, for all integers n where $n > 1$, that 3

$$\frac{4^n}{n+1} < \frac{(2n)!}{(n!)^2}$$

End of paper