SYDNEY BOYS HIGH SCHOOL
MOORE PARK, SURRY HILLS

2006
TRIAL HIGHER SCHOOL CERTIFICATE

Mathematics Extension 2

General Instructions
- Reading time – 5 minutes.
- Working time – 3 hours.
- Write using black or blue pen.
- Board approved calculators may be used.
- All *necessary* working should be shown in every question if full marks are to be awarded.
- Marks may **NOT** be awarded for messy or badly arranged work.
- Hand in your answer booklets in **3** sections.
  Section **A** (Questions 1 - 2),
  Section **B** (Questions 3 - 4)
  Section **C** (Questions 5 - 6)
  Section **D** (Questions 7 –8).
- Start each **NEW** section in a separate answer booklet.

Total Marks - 120 Marks
- Attempt Sections A - D
- All questions are of equal value.

Examiner: *E. Choy*

This is an assessment task only and does not necessarily reflect the content or format of the Higher School Certificate.

Total marks – 120
Attempt Questions 1 - 8
All questions are of equal value

Answer each section in a SEPARATE writing booklet. Extra writing booklets are available.

SECTION A (Use a SEPARATE writing booklet)

Question 1 (15 marks) **Marks**

(a) By first completing the square, evaluate the following integrals

(i) $\displaystyle\int_{-1}^{0} \frac{dx}{\sqrt{3-2x-x^2}}$ **2**

(ii) $\displaystyle\int_{0}^{1} \sqrt{x(1-x)}\, dx$ **2**

(b) Integrate the expressions below

(i) $\displaystyle\int \frac{1}{x\ln x}\, dx$ **1**

(ii) $\displaystyle\int x\ln x\, dx$ **2**

(iii) $\displaystyle\int \frac{x+1}{x^2+x+1}\, dx$ **2**

(c) Use the technique of *integration by parts* to evaluate **2**

$$\int_{0}^{\frac{1}{2}} \cos^{-1} x\, dx$$

(d) (i) Find real numbers A, B, and C so that **2**

$$\frac{10}{(3+x)(1+x^2)} = \frac{A}{3+x} + \frac{Bx+C}{1+x^2}$$

for all $x \neq -3$

(ii) Use part (i) above and the substitution $t = \tan\theta$ to find **2**

$$\int \frac{10d\theta}{3+\tan\theta}$$

SECTION A continued

Question 2 (15 marks) **Marks**

(a) (i) Write the complex number $-\sqrt{3}+i$ in modulus-argument form. 1

(ii) Hence, use de Moivre's Theorem to find $\left(-\sqrt{3}+i\right)^{10}$ in the form $a+ib$, for real values a and b. 2

(b) Sketch each of the following regions on separate Argand diagrams

(i) $-1<\operatorname{Re}(z)<2$ and $0<\operatorname{Im}(z)<3$ 2

(ii) $z\overline{z}-(1-i)z-(1+i)\overline{z}<2$ 2

(iii) $0<\arg\left[(1-i)z\right]<\frac{\pi}{6}$ 2

(c) (i) Find the square roots of the complex number $-3+4i$ 2

(ii) Find the roots of the quadratic equation $x^2-(4-2i)x+(6-8i)=0$ 2

(d) The locus of a point P, which moves in the complex plane, is represented by the equation $\left|z-(3+4i)\right|=5$

(i) Sketch the locus of the point P. 2

(ii) Find the modulus of z when $\arg z=\tan^{-1}\left(\frac{1}{2}\right)$. 2

SECTION B (Use a SEPARATE writing booklet)

Question 3 (15 marks)

(a) Find a cubic equation with roots α, β and γ such that 3

$$\left.\begin{aligned}\alpha\beta\gamma&=5\\ \alpha+\beta+\gamma&=7\\ \alpha^2+\beta^2+\gamma^2&=29\end{aligned}\right\}$$

(b) The polynomial $P(x)$ is defined by $P(x)=x^4+Ax^2+B$, where A and B are real positive numbers.

(i) Explain why $P(x)$ has no real zeroes. 2

(ii) If two of the zeroes of $P(x)$ are ib and $-id$ where b and d are real show that: 4

$$b^4+d^4=A^2-2B$$

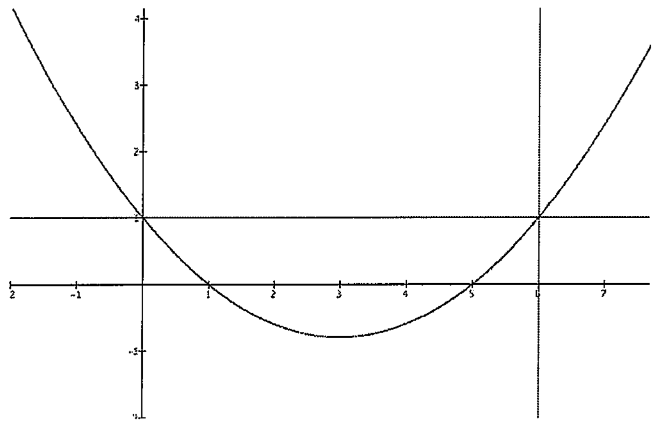
(c) Given that $f(x)=x^3-3ax+b$, where a and b are real numbers then:

(i) Show that $y=f(x)$ has turning points if $a>0$, and find their coordinates. 3

(ii) Show that $f(x)$ has three distinct real zeroes if $b^2<4a^3$. 3

Question 4 (15 marks) **Marks**

(a)



The sketch above shows the parabola $y = f(x)$, where

$$f(x) = \frac{1}{5}(x-1)(x-5).$$

Without any use of calculus, draw careful sketches of the following curves, showing all intercepts, asymptotes and turning points.

NB The vertex of the parabola is at $\left(3, -\frac{4}{5}\right)$.

(i) $y = \dfrac{1}{f(x)}$ **2**

(ii) $y = \left[f(x)\right]^2$ **3**

(iii) $y = \tan^{-1}\left[f(x)\right]$ **3**

(iv) $y = f(\ln x)$ **3**

(b) Suppose the function $f(x) = O(x) + E(x)$, where $O(x)$ is odd and $E(x)$ is even.

(i) By considering $f(-x)$, find an expression for $O(x)$ in terms of f. **2**

(ii) Hence write down $O(x)$ when $f(x) = e^x$. **2**

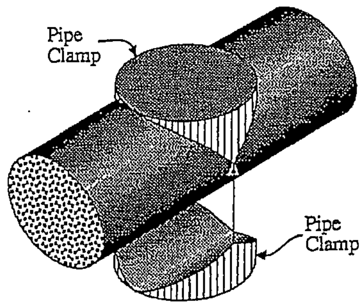
SECTION C starts on page 5

SECTION C (Use a SEPARATE writing booklet)

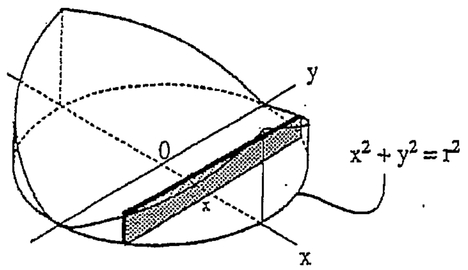
Question 5 (15 marks) **Marks**

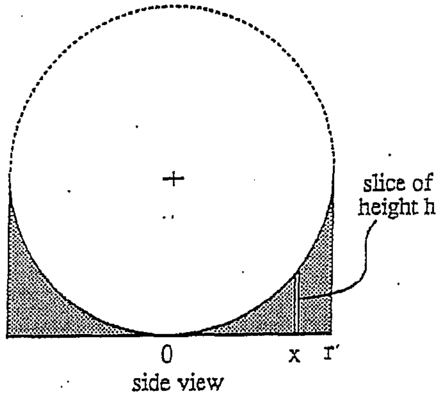
(a) A pipe-clamp is made of two identical pieces. Each piece has a circular base of radius r units and the other face is curved so as to fit flush against the pipe held between the two pieces.

The pipe also has a radius of r units.



A vertical slice, of thickness Δx, taken x units from the centre of the base is in the shape of a rectangle with one side in the circular base and of height necessary to reach the cylindrical pipe as shown in the diagram below:





(i) Show that the height of the slice taken x units from O is given by **3**

$$h = r - \sqrt{r^2 - x^2}$$

(ii) Show that the volume, ΔV, of such a slice is given by **3**

$$\Delta V \approx \left[2r\sqrt{r^2 - x^2} - 2\left(r^2 - x^2\right)\right]\Delta x$$

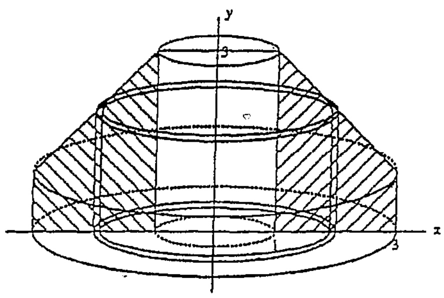
(iii) Hence, find by integration, the volume of ONE piece of the pipe-clamp. **3**

(b) (i) Show that the volume, ΔV, of a right cylindrical shell of height H, with inner radius r and thickness Δr is given by the formula **2**

$$\Delta V = 2\pi r H \Delta r$$

where Δr is sufficiently small so that $(\Delta r)^2$ may be neglected.

(b) (ii) A metal umbrella base is formed by rotating the area enclosed between $x = 1$, $x = 3$, $y = 0$ and $y = 4 - x$ about the y-axis as shown. **4**



Using the method of cylindrical shells, find the volume of the umbrella base.

Question 6 (15 marks)

(a) A point T moves so that the sum of its distances from the point $(-2,0)$ and $(2,0)$, on a Cartesian plane, is 6 units.

(i) Show that the locus of T is an ellipse $\mathscr{E}$ with the equation **2**

$$\frac{x^2}{9}+\frac{y^2}{5}=1$$

(ii) Find the equation of the auxiliary circle, $\mathscr{A}$, of $\mathscr{E}$. **1**

(iii) Find the eccentricity, coordinates of the foci and the equations of the directrices of the ellipse, $\mathscr{E}$. **2**

(iv) Draw a neat sketch, showing the ellipse and its auxiliary circle. **1**

(v) A line parallel to the y-axis meets the positive x-axis at N and the curves $\mathscr{E}$ and $\mathscr{A}$ at P and Q respectively. **1**
Given the coordinates $N(3\cos\theta,0)$, find the coordinates of P and Q (where P and Q are in the first quadrant).

(vi) Find the equations of the tangents at P and Q. **2**

(vii) If R is the point of intersection of the tangents at P and Q:

(α) Show that R lies on the major axis of $\mathscr{E}$. **2**

(β) Prove that the product of the lengths ON and OR is independent of the positions of P and Q on the curves. **2**

(b) Given p red balls and m yellow balls, where $p-m+1>0$, arranged in a row. Show that the number of ways of arranging them so that no two yellow balls appear together is given by: **2**

$$^{p+1}C_m$$

SECTION D starts on page 7

SECTION D (Use a SEPARATE writing booklet)

Question 7 (15 marks) **Marks**

(a) (i) Show that $z^5+1=(z+1)(z^4-z^3+z^2-z+1)$ **1**

(ii) If z is a solution to $z^5+1=0$ where $z \neq -1$, prove that $1+z^2+z^4=z+z^3$. **1**

(iii) Hence show that $\cos\frac{\pi}{5}+\cos\frac{3\pi}{5}=\frac{1}{2}$ **3**

(b) For integer values of k where $k=0, 1, 2, \ldots$ define I_k as follows:

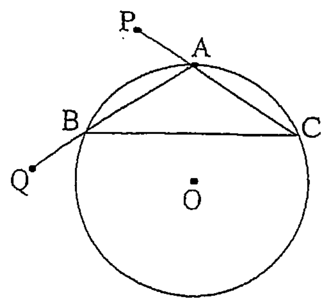
$$I_k=\int_0^{\frac{\pi}{2}} \cos^k x \, dx$$

(i) Express I_{k+2} in terms of k and I_k. **2**

(ii) Hence find an expression for I_{2n}, where $n=0, 1, 2, \ldots$ **2**

(c) In ΔABC, in the diagram on the right, $AB=AC$.

Produce CA to P and AB to Q so that $AP=BQ$.



(i) Show that $\angle OAP=\angle OBQ$. **3**

(ii) Prove that A, P, Q and O, the centre of circle ABC, are concyclic. **3**

Question 8 (15 marks)

(a) A particle is projected **vertically** upwards in a resisting medium where the resistance varies as the square of the velocity and k is the constant of variation. If the velocity of projection is $v_0 \tan\alpha$,

(i) Show that the maximum height, H, reached is given by: **3**

$$H=\frac{1}{2k}\ln\left(\frac{g+kv_0^2\tan^2\alpha}{g}\right)$$

(ii) Show that the particle returns to the point of projection with velocity $v_0\sin\alpha$ given that v_0 is the terminal velocity. **4**

(iii) Show that the time of ascent is $\frac{v_0\alpha}{g}$ **3**

(iv) Show that the time of descent is $\frac{v_0}{g}\ln(\sec\alpha+\tan\alpha)$ **2**

(b) Prove by induction that, for all integers n where $n>1$, that **3**

$$\frac{4^n}{n+1}<\frac{(2n)!}{(n!)^2}$$

End of paper