Question 1. (15 marks) Marks

(a) Find: (i) $\int \frac{1}{\sqrt{x+8}}\,dx$ 3

(ii) $\int \frac{1}{x^2+9}\,dx$

(b) Use integration by parts to find 3

$$\int x \ln x$$

(c) Use completion of squares to find 2

$$\int \frac{dx}{\sqrt{6-x-x^2}}$$

(d) i) Find real numbers a, b and c such that $\frac{1}{x^2(2-x)} = \frac{ax+b}{x^2} + \frac{c}{2-x}$ 4

ii) Hence evaluate $\int_1^{1.5} \frac{dx}{x^2(2-x)}$

(e) Use the substitution $x = \tan y$ to show that 3

$$\int_0^1 \frac{dx}{(x^2+1)^2} = \frac{\pi+2}{8}$$

$s^2 + c^2 = 1$
$t^2 + 1 = \sec^2$
$c2x = 2\cos^2 x - 1$
$\frac{c2x+1}{2} = co$

Question 2. (15 marks) Marks

(a) If k is a real number and $z = k - 2i$ express $\overline{(iz)}$ in the form $x + iy$ where x and y are real numbers. 2

(b) Solve the equation 2

$$\bar{z} = 3z - 1$$

where $z = x + iy$ $(x,\ y \text{ real})$

(c) On an Argand diagram shade the region specified by both the conditions 3

$$\text{Im}(z) \le 4 \text{ and } |z - 4 - 5i| \le 3$$

(d) If $\text{cis}\theta = \cos\theta + \text{i}\sin\theta$ express 2

$$(4\text{cis}\alpha)^2(2\text{cis}\beta)^3$$

in modulus-argument form.

(e) 4

i) If $w = \dfrac{z}{z+2}$ where $z = x + iy$ $(x,\ y \text{ real})$ find the locus of w given that it is purely imaginary.

ii) Sketch the locus of w on an Argand diagram.

(f) If α and β are real show that $(\alpha + \beta i)^{2002} + (\beta - \alpha i)^{2002} = 0$. 2

Question 3. (15 marks) Marks

(a) Consider the function 8

$$f(x)=\frac{x^3}{(1-x)^2}$$

i) Show that $f'(x)=\frac{x^2[3-x]}{(1-x)^3}$

ii) Use the first derivative $f'(x)$ to determine the nature of the stationary points.

iii) Write down the equations of any asymptotes.

iv) Sketch the graph of $y=f(x)$ showing all essential features.

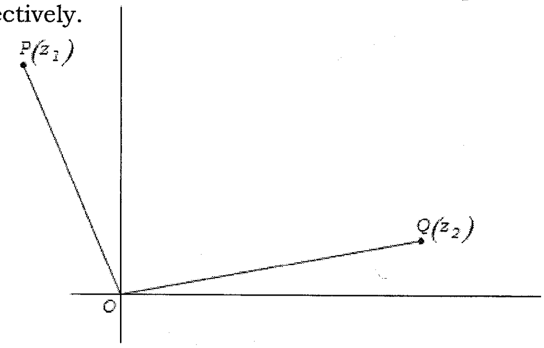
(b) 4

i) Sketch the graphs of $y=\sin x$ and $y=\sqrt{\sin x}$ for $0\le x\le\frac{\pi}{2}$ on the same diagram.

ii) Hence show that $1<\int_0^{\frac{\pi}{2}}\sqrt{\sin x}\,dx<\frac{\pi}{2}$

NOTE: You are NOT required to evaluate the integral $\int_0^{\frac{\pi}{2}}\sqrt{\sin x}\,dx$

(c) In the diagram below points P and Q represent the complex numbers z_1 and z_2 respectively. 3



i) Copy the diagram in your examination booklet and indicate the point representing the complex number z_1+z_2

ii) If the length of PQ is $|z_1-z_2|$ and $|z_1-z_2|=|z_1+z_2|$ what can be said about $\frac{z_2}{z_1}$

Question 4. (15 marks) Marks

(a) The real cubic polynomial $ax^3+9x^2+ax=30$ has $-3+i$ as a root. 4

i) Show that $x^2+6x+10$ is a quadratic factor of the cubic polynomial.

ii) Show that $a = 2$.

iii) Write down all the roots of the polynomial.

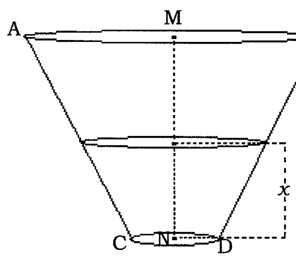
(b) Show that the polynomial $P(x)=nx^{n+1}-(n+1)x^n+1$ is divisible by $(x-1)^2$ 2

(c) i) Sketch the graphs of $y=\frac{1}{x^2+1}$ and $y=\frac{1}{x^2+2}$ on the same set of axes. 4

ii) The area bounded by the two curves in (i) and the ordinates at $x = 0$ and $x = 2$ is rotated about the y-axis. Use the cylindrical shell method to show that the volume of the resulting solid is

$$\pi \ln\frac{5}{3}.$$

(d) A drinking glass is in the shape of a truncated cone, in which the internal diameter of the top and bottom are 8cm and 6cm respectively. 5



AB = 8cm
CD = 6cm

i) If the internal height of the glass, MN, is 10cm show that the area of the cross-section x cm above the base is

$$\pi\left(3+\frac{x}{10}\right)^2 \text{ cm}^2.$$

ii) Hence find by integration, the volume of liquid the glass can hold (answer to the nearest mL).

Question 5. (15 marks) Marks

The equation of an ellipse E is given by $\frac{x^2}{9}+\frac{y^2}{5}=1$

i) Find the eccentricity of E — 1

ii) Write down the — 3

α) coordinates of the foci

β) equations of the directrices

γ) equation of the major auxiliary circle A.

iii) Draw a neat sketch of E showing clearly the features in part ii) — 2

iv) A line parallel to the positive y-axis meets the x-axis at N and the curves E, A at P and Q respectively. If N has coordinates $(3\cos\theta, 0)$ find the coordinates of P and Q. [P and Q are in the first quadrant] — 2

v) Show that the equations of the tangents at P and Q are $\sqrt{5}x\cos\theta + 3y\sin\theta = 3\sqrt{5}$ and $x\cos\theta + y\sin\theta = 3$ respectively. — 4

vi) Show that the point of intersection R of these tangents lies on the major axis of E produced. — 1

vii) Prove that $ON \cdot OR$ is independent of the position of P and Q on the curves. — 2

Question 6. (15 marks) Marks

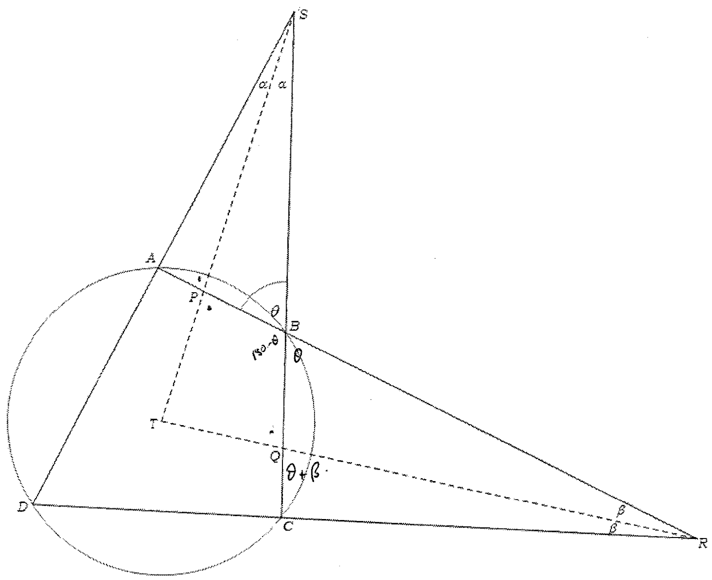
(a) i) A particle of mass m falls vertically from rest, from a point o, in a medium whose resistance is mkv, where k is a positive constant and v its velocity after t seconds. 4

Show that $v = \frac{g}{k}\left(1 - e^{-kt}\right)$

ii) An equal particle is projected vertically upwards with initial velocity U in the same medium. [The particle is released simultaneously with the first particle]. 4

Show that the velocity of the first particle when the second particle is momentarily at rest is given by $\dfrac{VU}{V+U}$ where V is the terminal velocity of the first particle.

(b) 7



$ABCD$ is a cyclic quadrilateral.

The sides AB and CD produced intersect at R and the sides CB and DA produced intersect at S. ST and RT intersect AR and CS at P and Q respectively.

The bisectors of $C\hat{S}D$ and $A\hat{R}D$ meet at T.

Let $A\hat{S}T = B\hat{S}T = \alpha$ and $A\hat{R}T = D\hat{R}T = \beta$ and $A\hat{B}S = \theta$.

i) Show that $T\hat{P}B + T\hat{Q}B = \alpha + \beta + 2\theta$

ii) Prove that ST is perpendicular to RT.

Question 7. (15 marks) Marks

(a) Given that $\tan 5\theta = \dfrac{t^5 - 10t^3 + 5t}{5t^4 - 10t^2 + 1}$, where $t = \tan\theta$ [Do not prove this] 5

i) Solve the equation $\tan 5\theta = 0$ for $0 \le \theta \le \pi$

ii) Hence prove that

α) $\tan\dfrac{\pi}{5}\tan\dfrac{2\pi}{5} = \sqrt{5}$

β) $\tan^2\dfrac{\pi}{5} + \tan^2\dfrac{2\pi}{5} = 10$

(b) i) Show that $\int x\tan^{-1}x\,dx = \dfrac{1}{2}(x^2+1)\tan^{-1}x - \dfrac{1}{2}x + c$ 4

ii) If $u_n = \int_0^1 x^n \tan^{-1}x\,dx$ for $n \ge 2$ show that

$$u_n = \frac{\pi}{2(n+1)} - \frac{1}{n(n+1)} - \frac{n-1}{n+1}u_{n-2}$$

(c) Show that the number of ways in which $2n$ persons may be seated at two round tables, n persons being seated at each is 2

$$\frac{(2n)!}{n^2}$$

(d) i) There are 6 persons from whom a game of tennis is to be made up, two on each side. How many different matches can be arranged if a change in either pair gives a different match? 4

ii) How many different matches are possible if two particular persons are to both play in the match?

Question 8. (15 marks) Marks

(a) Suppose a, b, c and d are positive real numbers. 5

i) Prove that $\frac{a}{b}+\frac{b}{a}\geq 2$.

ii) Deduce that $\frac{a+b+c}{d}+\frac{b+c+d}{a}+\frac{c+d+a}{b}+\frac{d+a+b}{c}\geq 12$.

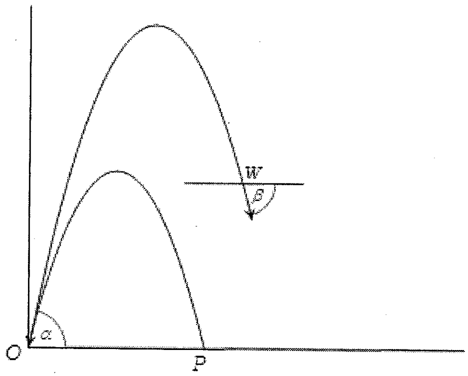
iii) Hence prove that if $a+b+c+d=1$, then:

$$\frac{1}{a}+\frac{1}{b}+\frac{1}{c}+\frac{1}{d}\geq 16.$$

(b) Two stones are thrown simultaneously from the same point O in the same direction and with the same non-zero angle of projection α, but with different velocities U and V ($U < V$). 6

The slower stone hits the ground at a point P on the same level as the point of projection.

At that instant the faster stone is at a point W on its downward path, making an angle β with the horizontal.



i) Show that $V(\tan\alpha+\tan\beta)=2U\tan\alpha$

ii) Deduce that if $\beta=\frac{1}{2}\alpha$ then $U<\frac{3}{4}V$

(c) i) Show by graphical means that 4

$$\ln ex > e^{-x} \text{ for } x\geq 1$$

ii) Hence, or otherwise, show that

$$\ln(n!e^{n}) > e^{-n}\left(\frac{e^{n}-1}{e-1}\right)$$