SCEGGS Darlinghurst

2008

HIGHER SCHOOL CERTIFICATE
TRIAL EXAMINATION

Mathematics Extension 2

This is a TRIAL PAPER only and does not necessarily reflect the content or format of the Higher School Certificate Examination for this subject.

General Instructions

- Reading time – 5 minutes
- Working time – 3 hours
- Write using black or blue pen
- Board-approved calculators may be used
- A table of standard integrals is provided at the back of this paper
- All necessary working should be shown in every question

Total marks – 120

- Attempt Questions 1–8
- All questions are of equal value

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All questions are of equal value

Answer each question in a SEPARATE writing booklet. Extra writing booklets are available.

Marks

Question 1 (15 marks)

(a) Find $\displaystyle\int \frac{dx}{\sqrt{16x^2 - 1}}$ **2**

(b) Evaluate $\displaystyle\int_1^e x \ln x \, dx$ **3**

(c) (i) Find real numbers a and b such that **2**

$$\frac{5x^2 + x + 8}{(x+1)(x^2+3)} \equiv \frac{a}{x+1} + \frac{bx - 1}{x^2 + 3}$$

(ii) Hence find $\displaystyle\int \frac{5x^2 + x + 8}{(x+1)(x^2+3)}\, dx$ **2**

(d) Find $\displaystyle\int \tan^3 x \, dx$ **2**

(e) Using a suitable substitution, or otherwise, evaluate: **4**

$$\int_0^2 \frac{x^2}{\sqrt{4 - x^2}}\, dx$$

End of Question 1

Marks

Question 2 (15 marks) Use a SEPARATE writing booklet.

(a) Let $\alpha = 1 - \sqrt{3}\,i$.

(i) Find the exact value of $|\alpha|$ and $\arg\alpha$. **2**

(ii) Hence express $\left(1-\sqrt{3}\,i\right)^{10}$ in modulus-argument form. **1**

(b) Express $\sqrt{7-24i}$ in the form $a+ib$, where a and b are real. **3**

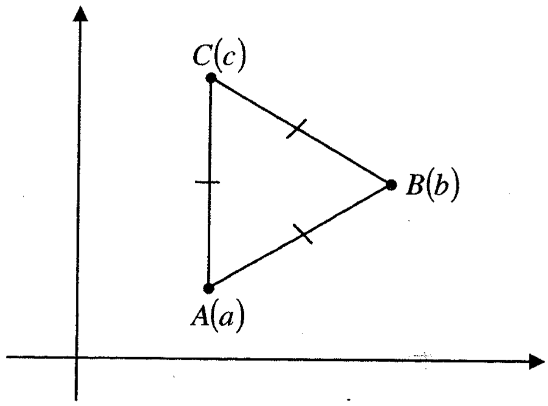
(c) Sketch the region in the complex plane where the two inequalities $0 \le \operatorname{Arg}(z) \le \dfrac{3\pi}{4}$ and $|z-2i| \ge |z|$ both hold. **3**

(d) Sketch the locus of z satisfying $|z-3| + |z+3| = 10$. Show any intercepts with the axes. **3**

Question 2 continues on page 4

Marks

(e)



The points A, B and C on the Argand diagram represent the complex numbers a, b and c respectively. ΔABC is equilateral.

Let $w = \cos\frac{\pi}{3} + i\sin\frac{\pi}{3}$.

(i) Show that $\dfrac{a-b}{c-b} = w$. **1**

(ii) By writing another similar expression for w, prove that **2**

$$a^2 + b^2 + c^2 = ab + bc + ca$$

End of Question 2

Question 3 (15 marks) Use a SEPARATE writing booklet. **Marks**

(a) The equation $x^3 + 3x^2 - 5x - 2 = 0$ has roots α, β and γ. **2**

Find a cubic equation with integer coefficients whose roots are $\frac{2}{\alpha}, \frac{2}{\beta}$ and $\frac{2}{\gamma}$.

(b) Consider the curve $x^2 + y^2 + xy = 3$.

(i) Show that $\frac{dy}{dx} = -\left(\frac{2x + y}{x + 2y}\right)$. **1**

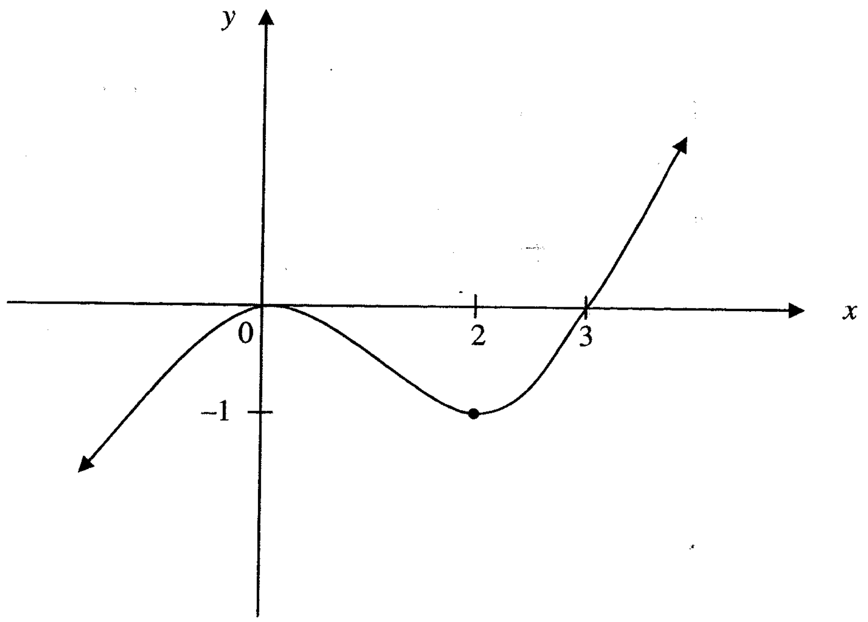
(ii) Hence find the coordinates of any stationary points. **2**

Question 3 continues on page 6

Marks

Question 3 (continued)

(c) The diagram shows the graph of $y = f(x)$ where $f(x) = \frac{1}{4}x^2(x-3)$.



On the answer page provided, draw separate sketches of the graphs of the following:

(i) $y = \frac{1}{4}x^2 \mid x - 3 \mid$ **1**

(ii) $y = \dfrac{1}{f(x)}$ **1**

(iii) $y^2 = -f(x)$ **2**

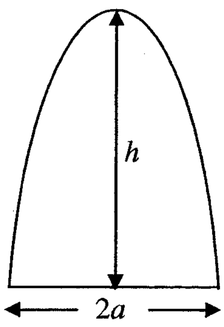
(iv) $y = \tan^{-1}(f(x))$ **2**

Question 3 continues on page 7

Marks

Question 3 (continued)

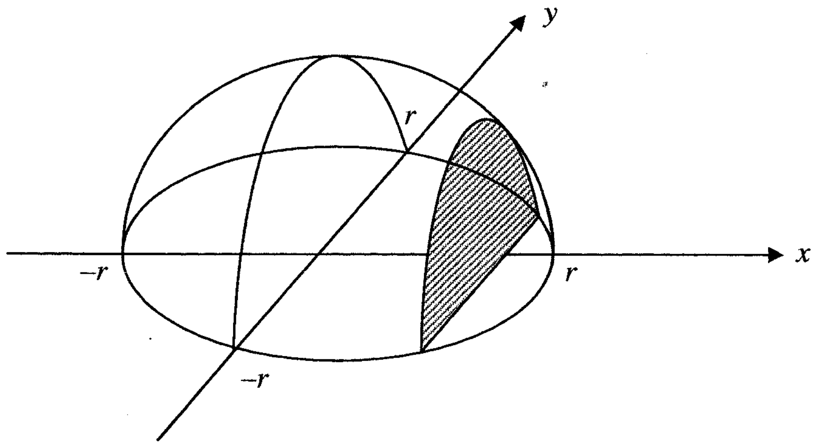
(d) (i) **1**



A parabolic segment has height h and width $2a$.

Use Simpson's Rule with three function values, to show that the exact area of this segment is $\frac{4ah}{3}$.

(ii) **3**



The base of a solid is the region in the xy plane enclosed by the circle $x^2 + y^2 = r^2$.

Each cross-section perpendicular to the x-axis is a parabolic segment with height one half its width.

Show that the volume of the solid is $\frac{16r^3}{9}$ units^3.

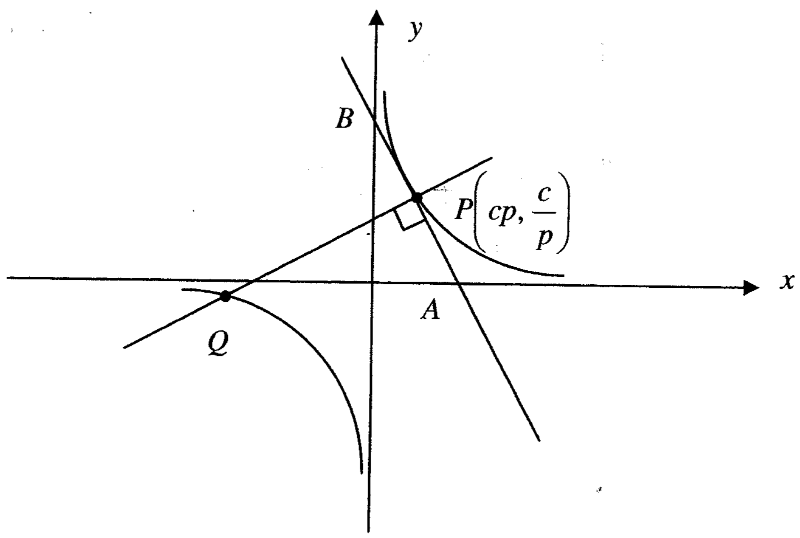
End of Question 3

Marks

Question 4 (15 marks) Use a SEPARATE writing booklet.

(a) The point $P\left(cp, \frac{c}{p}\right)$ is a point on the hyperbola $xy = c^2$.

The tangent to the hyperbola at P intersects the x and y axes at A and B respectively and the normal to the hyperbola at P intersects the second branch at Q.



(i) Show that the equation of the normal at P is $py - c = p^3(x - cp)$. **2**

(ii) Show that the x coordinates of P and Q satisfy the equation **2**

$$x^2 - c\left(p - \frac{1}{p^3}\right)x - \frac{c^2}{p^2} = 0$$

and hence find the coordinates of Q.

(iii) Given the distance $AB = 2c\sqrt{p^2 + \frac{1}{p^2}}$, show that the **2**

area of $\Delta ABQ = c^2\left(p^2 + \frac{1}{p^2}\right)^2$.

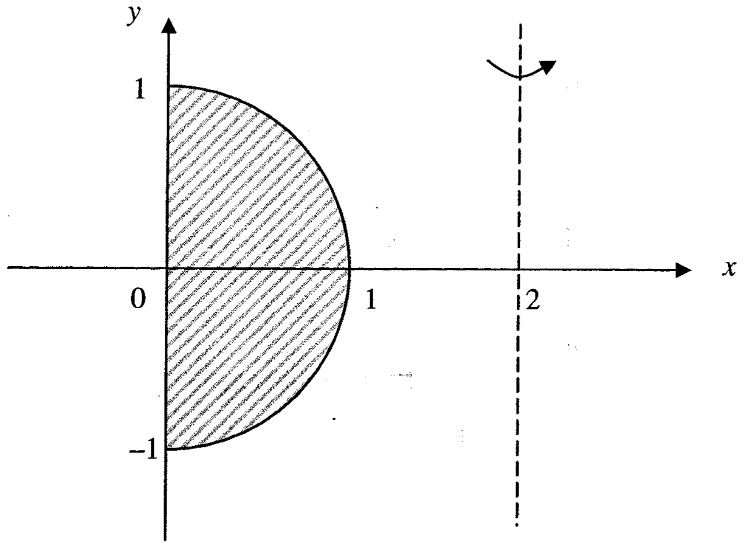
(iv) Find the minimum area of ΔABQ. **1**

(You may use the inequality $\frac{a}{b} + \frac{b}{a} \geq 2$ for $a, b > 0$.)

Question 4 continues on page 9

Marks

(b)



The shaded semicircle in the diagram above is rotated about the line $x = 2$.

(i) Using the method of cylindrical shells, show that the volume V of the resulting solid is given by **1**

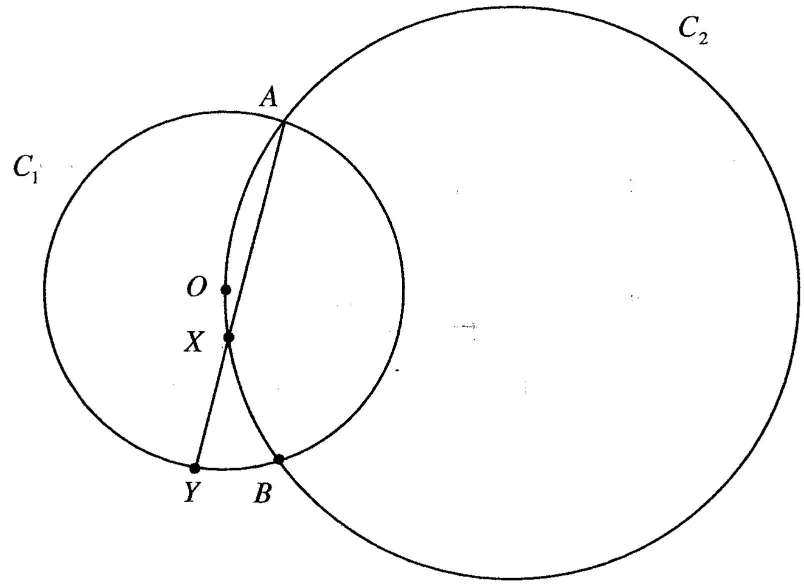
$$V = \int_0^1 4\pi\,(2 - x)\sqrt{1 - x^2}\,dx$$

(ii) Hence find the volume of the solid. **3**

Question 4 continues on page 10

Question 4 (continued) **Marks**

(c)



Two circles C_1 and C_2 intersect at A and B. C_2 passes through O, the centre of C_1. X lies on the arc AOB and AX intersects C_1 again at Y.

(i) State why $\angle AOB = 2 \times \angle AYB$. **1**

(ii) Prove that $XY = XB$. **3**

End of Question 4

Marks

Question 5 (15 marks) Use a SEPARATE writing booklet.

(a) (i) Show that if α is a double root of $f(x)=0$ then $f(\alpha)=f'(\alpha)=0$. **2**

(ii) Find all roots of the equation $2x^3-5x^2-4x+12=0$ given that two of the roots are equal. **3**

(b) (i) By drawing a diagram, or otherwise, find the solutions of $z^5=1$. **2**

(ii) Show that $\cos\frac{2\pi}{5}+\cos\frac{4\pi}{5}=-\frac{1}{2}$. **2**

(iii) Hence find the exact value of $\cos\frac{2\pi}{5}$. **2**

(c) 11 persons gather to play basketball by forming 2 teams of 5 to play each other. The remaining person acts as a referee.

(i) In how many ways can the teams be formed? **2**

(ii) If two particular persons are not to be in the same team, how many ways are there then to choose the teams? **2**

End of Question 5

Marks

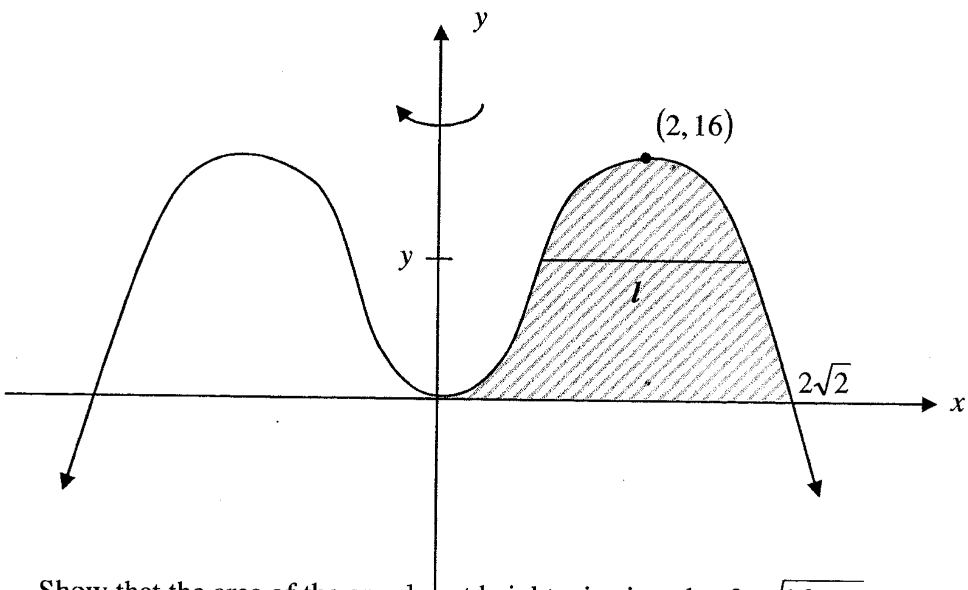
Question 6 (15 marks) Use a SEPARATE writing booklet.

(a) The sequence $\{a_n\}$ is given by: **3**

$$a_1 = 2,\ a_2 = \frac{3}{2} \text{ and } (n+1)\,a_{n+1} = a_{n-1} - (n-2)a_n \text{ for } n > 1.$$

Prove by induction that for $n \geq 1$, $a_n = \dfrac{n+1}{n!}$

(b) The region bound by the curve $y = 8x^2 - x^4$ and the x axis in the first quadrant is rotated about the y axis to form a solid. When the region is rotated, the horizontal line segment l at height y sweeps out an annulus.



(i) Show that the area of the annulus at height y is given by $2\pi\sqrt{16-y}$. **3**

(ii) Find the volume of the solid. **2**

Question 6 continues on page 13

Marks

Question 6 (continued)

(c) (i) Differentiate $x\cos^{n-1}x$. **1**

(ii) Let $I_n = \int_0^{\frac{\pi}{2}} x\cos^n x \, dx$ for $n = 0, 1, 2, \cdots$ **4**

Show that for $n \geq 2$

$$I_n = -\frac{1}{n^2} + \frac{n-1}{n} I_{n-2}$$

(iii) Hence evaluate $\int_0^{\frac{\pi}{2}} x\cos^4 x \, dx$. **2**

End of Question 6

Marks

Question 7 (15 marks) Use a SEPARATE writing booklet.

(a) (i) If $z = \cos\theta + i\sin\theta$, show that $z + \dfrac{1}{z} = 2\cos\theta$ and $z^n + \dfrac{1}{z^n} = 2\cos n\theta$. **2**

(ii) Hence show that $\cos^5\theta = \dfrac{1}{16}\cos 5\theta + \dfrac{5}{16}\cos 3\theta + \dfrac{5}{8}\cos\theta$. **2**

(iii) Hence find the general solution to the equation **3**

$$16\cos^5\theta = 15\cos 3\theta + \cos 5\theta.$$

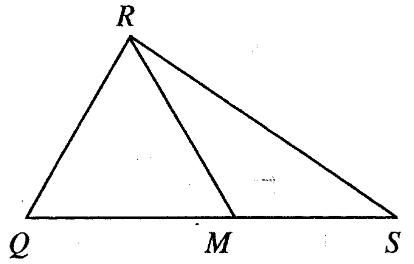
Question 7 continues on page 15

Question 7 (continued)

For parts (b) and (c) you may use the following identity:

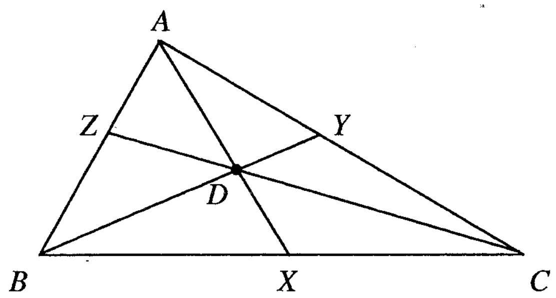
$$\text{If } \frac{P}{Q} = \frac{R}{S}, \text{ then } \frac{P}{Q} = \frac{R}{S} = \frac{P \pm R}{Q \pm S}$$

(b) (i) **1**



Show that $\dfrac{QM}{MS} = \dfrac{\text{Area } \Delta RQM}{\text{Area } \Delta RMS}$.

(ii)



In the diagram, *Z*, *X* and *Y* lie on the sides of ΔABC *AB*, *BC* and *CA* respectively such that *AX*, *BY* and *CZ* are concurrent.
D is the point of concurrency.

(α) Show that $\dfrac{BX}{XC} = \dfrac{\text{Area } \Delta ABD}{\text{Area } \Delta ACD}$. **2**

(β) Hence prove $\dfrac{BX}{XC} \cdot \dfrac{CY}{YA} \cdot \dfrac{AZ}{ZB} = 1$. **2**

Question 7 continues on page 16

Question 7 (continued)

Marks

(c) a, x, y, z are real numbers such that

$$\frac{\cos x + \cos y + \cos z}{\cos(x+y+z)} = \frac{\sin x + \sin y + \sin z}{\sin(x+y+z)} = a$$

(i) Use the identity given earlier to show that **1**

$$a = \frac{cisx + cisy + cisz}{cis(x+y+z)}$$

(ii) Hence show that **2**

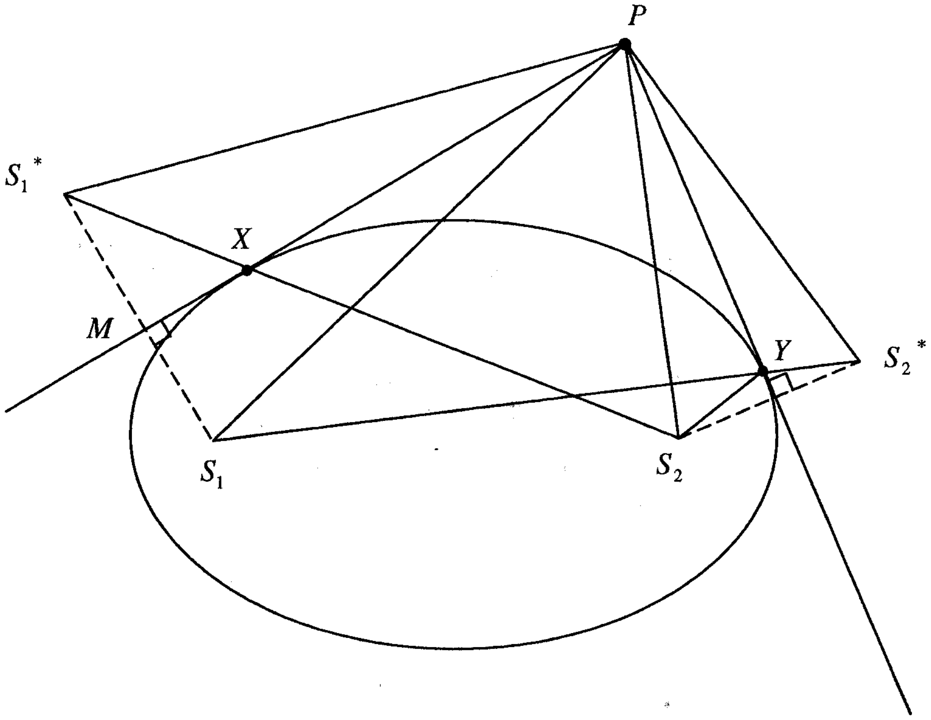
$$a = \cos(y+z) + \cos(x+z) + \cos(x+y)$$

End of Question 7

Marks

Question 8 (15 marks) Use a SEPARATE writing booklet.

(a)



In the diagram, X and Y are arbitrary points on the ellipse and tangents to the ellipse at X and Y meet at the point P. The points S_1 and S_2 are the foci of the ellipse, and S_1^* and S_2^* are the reflections of S_1 and S_2 across the tangents, as shown. $S_1 S_1^*$ and the tangent at X intersect at the point M.

You may assume, without proof, the following two properties of an ellipse:

1. The sum of the focal lengths from any point on an ellipse is constant.
2. The reflection property:
   Tangents to an ellipse are equally inclined to the focal chords drawn through the point of contact.

(i) Prove $\Delta MXS_1 \equiv \Delta MXS_1^*$ and hence show that $S_1^* XS_2$ is a straight line. [Note that similarly, $S_1 YS_2^*$ is a straight line.] **3**

(ii) Prove that $S_1^* S_2 = S_1S_2^*$ **2**

(iii) Hence state why $\Delta S_1^* PS_2 \equiv \Delta S_1PS_2^*$. **1**

(iv) Deduce that $\angle S_1PX = \angle S_2PY$. **2**

Question 8 continues on page 18

Marks

Question 8 (continued)

(b) (i) What value of x maximizes the expression $\log_e x - x + 1$? **1**

(ii) Deduce that $\log_e x \le x - 1$ for $x > 0$. **1**

(iii) Consider the set of n positive numbers **2**

$$p_1, p_2, \ldots, p_n \text{ such that } p_1 + p_2 + \ldots + p_n = 1.$$

Use the result in part (ii) to show that

$$\log_e(np_1) + \log_e(np_2) + \ldots + \log_e(np_n) \le 0$$

(iv) Deduce that $n^n p_1 p_2 \ldots p_n \le 1$. **1**

(v) Let $A = x_1 + x_2 + \ldots + x_n \quad (x_1, x_2, \ldots, x_n \ge 0)$ and set **1**

$p_1 = \frac{x_1}{A},\ p_2 = \frac{x_2}{A}, \ldots, p_n = \frac{x_n}{A}$.

Prove that $\frac{x_1 + x_2 + \ldots + x_n}{n} \ge \sqrt[n]{x_1\, x_2 \ldots x_n}$.

(vi) Show that for $a, b, c, d > 0$, with $abcd = 1$ **1**

$a^2 + b^2 + c^2 + d^2 + ab + ac + ad + bc + bd + cd \ge 10$.

End of Paper

2008 HIGHER SCHOOL CERTIFICATE TRIAL EXAMINATION

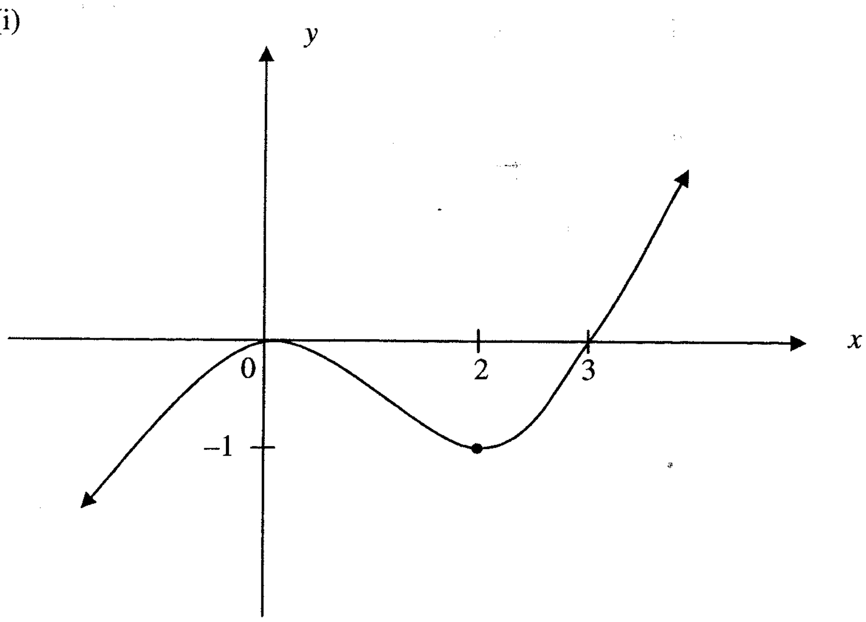
Mathematics Extension 2

Centre Number

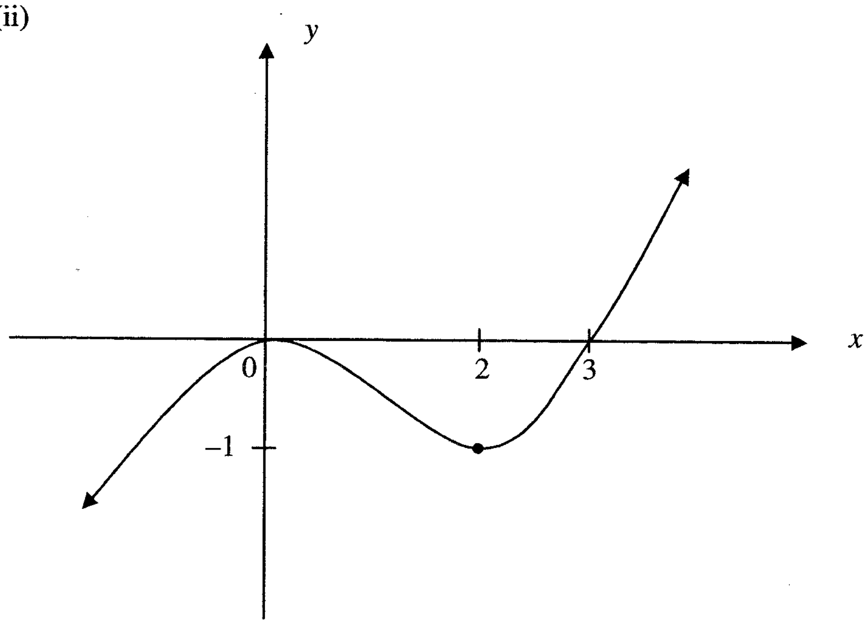
Student Number

Questions 3 (c)

(i)



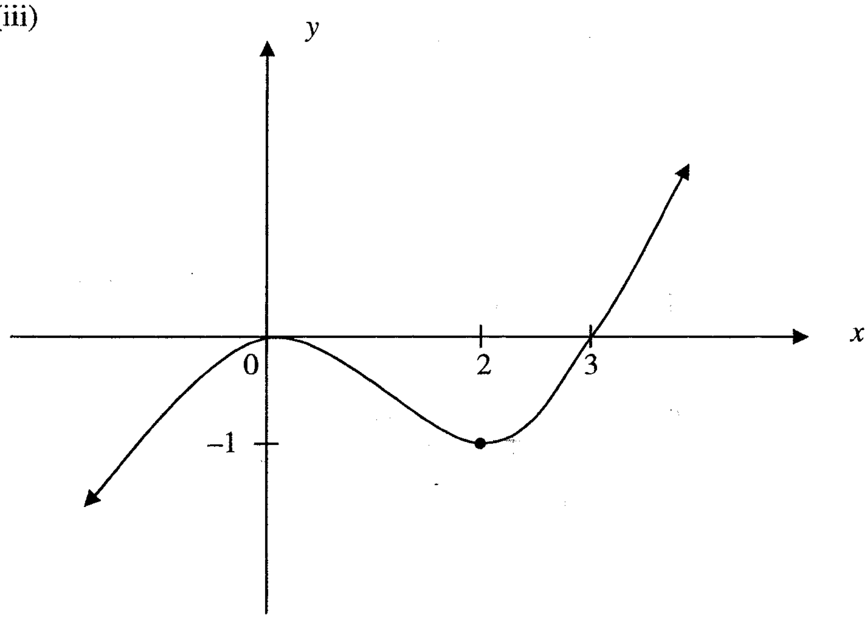
(ii)



Question 3(c) continues on next page

Question 3 (continued)

(c) (iii)



(iv)