SCEGGS Darlinghurst

2009

HIGHER SCHOOL CERTIFICATE
TRIAL EXAMINATION

Mathematics Extension 2

This is a TRIAL PAPER only and does not necessarily reflect the content or format of the Higher School Certificate Examination for this subject.

General Instructions

- Reading time – 5 minutes
- Working time – 3 hours
- Write using black or blue pen
- Board-approved calculators may be used
- A table of standard integrals is provided at the back of this paper
- All necessary working should be shown in every question

Total marks – 120

- Attempt Questions 1–8
- All questions are of equal value

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Total marks – 120
Attempt Questions 1–8
All questions are of equal value

Answer each question in a SEPARATE writing booklet. Extra writing booklets are available.

Marks

Question 1 (15 marks)

(a) Find $\displaystyle\int_0^{\frac{\pi}{3}} \sec^3 x \tan x \; dx$ **2**

(b) Find $\displaystyle\int \sqrt{\frac{5-x}{5+x}} \; dx$. **3**

(c) (i) Find real numbers A, B and C such that **2**

$$\frac{10}{(3+x)\left(1+x^2\right)} \equiv \frac{A}{3+x} + \frac{Bx+C}{1+x^2}.$$

(ii) Use the substitution $t = \tan\theta$ to find $\displaystyle\int \frac{10}{3+\tan\theta} \; d\theta$. **3**

(d) For $n \geq 1$, let $I_n = \displaystyle\int_0^1 \frac{dx}{\left(x^2+1\right)^n}$.

(i) By writing $\displaystyle\int_0^1 \frac{dx}{\left(x^2+1\right)^n}$ as $\displaystyle\int_0^1 1 \times \frac{dx}{\left(x^2+1\right)^n}$, and using integration by parts, **3**

show that

$$2nI_{n+1} = 2^{-n} + (2n-1)\, I_n .$$

(ii) Hence evaluate $\displaystyle\int_0^1 \frac{dx}{\left(x^2+1\right)^3}$. **2**

End of Question 1

Marks

Question 2 (15 marks) Use a SEPARATE writing booklet.

(a) The complex number w is given by $w = -1 + \sqrt{3}\, i$.

(i) Show that $w^2 = 2\overline{w}$. **1**

(ii) Evaluate $|\, w \,|$ and $\arg w$. **2**

(iii) Show that w is a root of $z^3 - 8 = 0$. **1**

(b) On separate diagrams, draw a neat sketch of the locus defined by

(i) $|\, z - 1 - 3i \,| \le 2$ and $\dfrac{\pi}{4} \le \arg z \le \dfrac{\pi}{2}$. **2**

(ii) $\arg\left(\dfrac{z - i}{z - 1}\right) = \dfrac{\pi}{2}$. **2**

(c) By considering the binomial expansion of $(1 + i)^n$ show that **3**

$$1 - \binom{n}{2} + \binom{n}{4} - \binom{n}{6} + \cdots = 2^{\frac{n}{2}} \cos \frac{n\pi}{4}.$$

(d) The points O, I, Z and P on the Argand Plane represent the complex numbers 0, 1, z and z + 1 respectively, where $z = \cos\theta + i \sin\theta$ is any complex number of modulus 1, with $0 < \theta < \pi$.

(i) Explain why *OIPZ* is a rhombus. **1**

(ii) Show that $\dfrac{z - 1}{z + 1}$ is purely imaginary. **2**

(iii) Find the modulus of $z + 1$ in terms of θ. **1**

End of Question 2

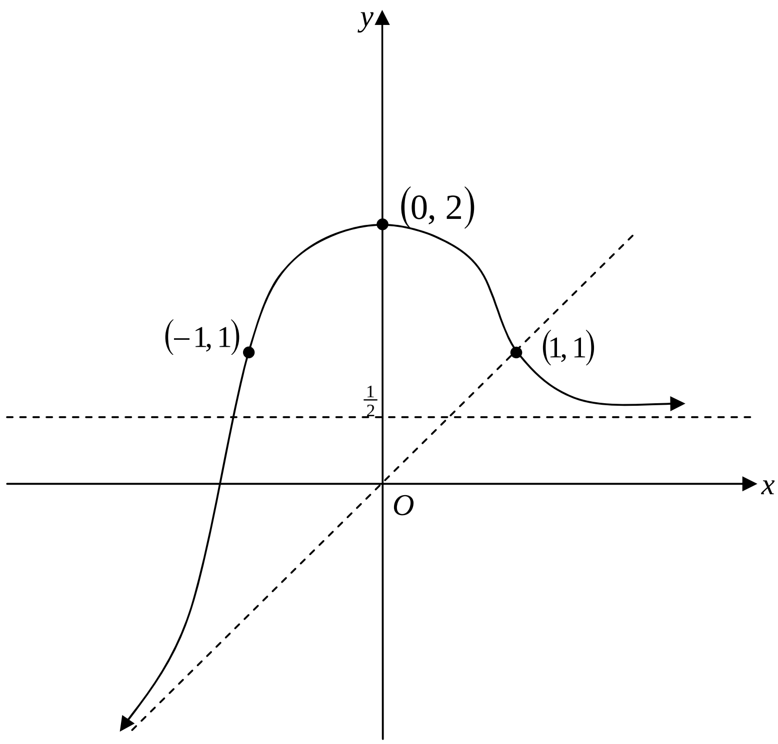
Marks

Question 3 (15 marks) Use a SEPARATE writing booklet.

(a) The locus defined by $|z-2|-|z+2|=2$ corresponds to part of a hyperbola in the Argand Plane. **3**

Sketch the locus labeling the foci, directrices, asymptotes and any intercepts with the axes.

(b)



The diagram shows the graph of $y=f(x)$. The lines $y=x$ and $y=\frac{1}{2}$ are both asymptotes.

On the answer page provided, draw separate sketches of the following graphs. Clearly indicate any important features.

(i) $y=(f(x))^2$ **2**

(ii) $y=\dfrac{1}{f(x)}$ **2**

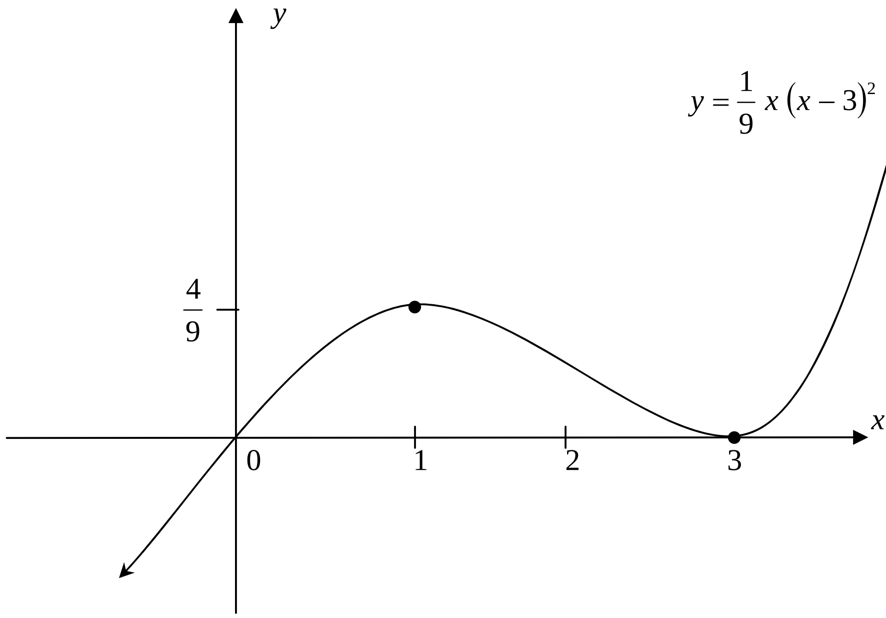
(iii) $y=f(x)-x$ **2**

Question 3 continues on page 5

Marks

Question 3 (continued)

(c)



(i) Given the sketch of $y=\frac{1}{9}x(x-3)^2$ above, sketch the curve $y^2=\frac{1}{9}x(x-3)^2$. **2**

(ii) Use implicit differentiation to find $\frac{dy}{dx}$ in terms of x and y for $y^2=\frac{1}{9}x(x-3)^2$. **1**

(iii) Given that the length of a curve between $x=a$ and $x=b$ is given by **3**

$$\int_a^b \sqrt{1+\left(\frac{dy}{dx}\right)^2}\ dx\,,$$

find the entire length of the curve $y^2=\frac{1}{9}x(x-3)^2$ for $0\le x\le 3$.

End of Question 3

Marks

Question 4 (15 marks) Use a SEPARATE writing booklet.

(a) Consider the polynomial $P(x) = x^4 - 4x^3 + 11x^2 - 14x + 10$.

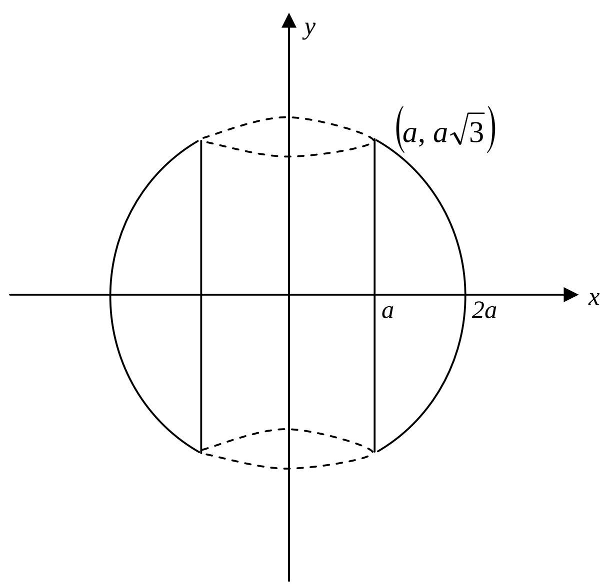
(i) If $P(x)$ has zeroes $a + bi$ and $a + 2bi$, where a and b are real and $b > 0$, find the values of a and b. **3**

(ii) Hence express $P(x)$ as the product of two quadratic factors with real coefficients. **1**

(b) The region bounded by the curve $y = \cos x$ and the coordinate axes is rotated about the y-axis. **3**

Use the method of cylindrical shells to find the volume of the solid formed.

(c) **3**



A cylindrical hole of radius a cm is bored through the centre of a sphere of radius $2a$ cm.

Show that the volume of the remaining solid is $4\sqrt{3}a^3\pi$ cm^3.

Question 4 continues on page 7

Marks

Question 4 (continued)

(d) The cubic equation $x^3 + kx + 1 = 0$, where k is a constant, has roots α, β and γ.
For each positive integer n, $S_n = \alpha^n + \beta^n + \gamma^n$.

(i) State the value of S_1 and express S_2 in terms of k. **2**

(ii) Show that for all n, $S_{n+3} + kS_{n+1} + S_n = 0$. **2**

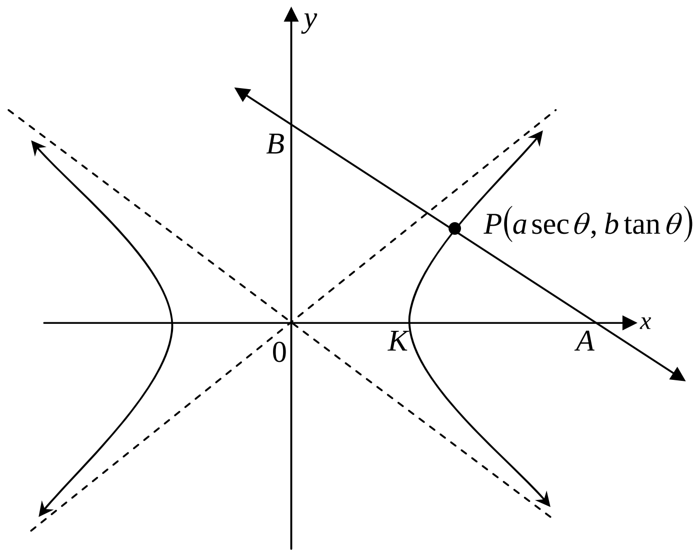
(iii) Hence, or otherwise, express $\alpha^4 + \beta^4 + \gamma^4$ in terms of k. **1**

End of Question 4

Marks

Question 5 (15 marks) Use a SEPARATE writing booklet.

(a)



The hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$, where $a > b > 0$, cuts the positive x–axis at the point K.

The normal to the hyperbola at the point $P(a\sec\theta, b\tan\theta)$ cuts the x–axis at A and the y–axis at B, as shown in the diagram.

(i) Show that the equation of the normal to the hyperbola at the point P is **2**

$$\frac{ax}{\sec\theta} + \frac{by}{\tan\theta} = a^2 + b^2.$$

(ii) Find the midpoint M of AB. **3**

(iii) Find the point G such that G divides the interval OM in the ratio 2:1. **1**

(iv) Show that the locus of G is a hyperbola and find the point L at which this locus cuts the positive x–axis. **3**

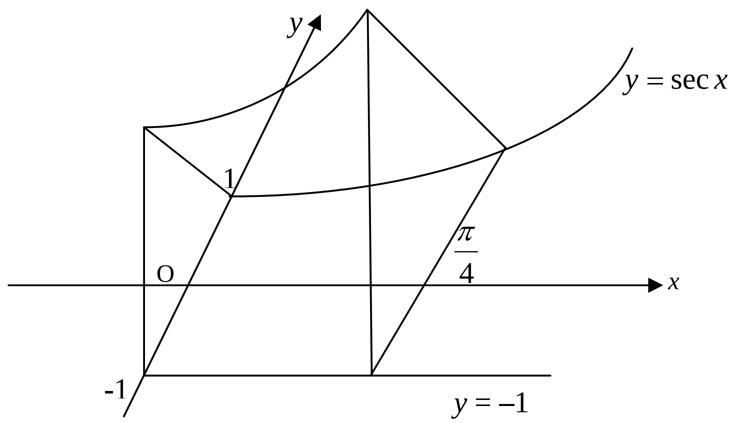
(v) If $\frac{OL}{OK} < 1$, show that $1 < e < \sqrt{3}$. **2**

Question 5 continues on page 9

Marks

Question 5 (continued)

(b) The base of a solid is the region in the xy plane enclosed by the curve $y = \sec x$ and $y = -1$ for $0 \le x \le \frac{\pi}{4}$. Each cross-section perpendicular to the x–axis is an equilateral triangle.



(i) Show that the area of the triangular cross-section at a distance x from the y – axis is $\frac{\sqrt{3}}{4}(\sec x + 1)^2$. **1**

(ii) Hence find the volume of the solid. **3**

End of Question 5

Marks

Question 6 (15 marks) Use a SEPARATE writing booklet.

(a) Let $I_1 = \int_0^{\pi} \frac{x \sin x}{1 + \cos^2 x}\, dx$ and let $I_2 = \int_0^{\pi} \frac{(\pi - x) \sin x}{1 + \cos^2 x}\, dx$

(i) Using a suitable substitution show that $I_1 = I_2$. **1**

(ii) Find the value of $I_1 + I_2$ and hence evaluate $\int_0^{\pi} \frac{x \sin x}{1 + \cos^2 x}\, dx$. **3**

(b) Let $z = \cos\theta + i \sin\theta$ be any complex number of modulus 1.

(i) Show that $\frac{z^2 - 1}{z} = 2i \sin\theta$. **2**

(ii) Using the formula for the sum of a Geometric Progression and the result in part (i), prove that **2**

$$z + z^3 + z^5 + z^7 + z^9 = \frac{\sin 10\theta + i(1 - \cos 10\theta)}{2 \sin\theta}.$$

(iii) Hence write down a simplified expression for **3**

$$\cos\theta + \cos 3\theta + \cos 5\theta + \cos 7\theta + \cos 9\theta$$

and find the general solution to

$$\cos\theta + \cos 3\theta + \cos 5\theta + \cos 7\theta + \cos 9\theta = \frac{1}{2}.$$

Question 6 continues on page 11

Marks

Question 6 (continued)

(c) Seven players are entered in a round robin tennis competition. Each round consists of three singles matches with the 7th player obtaining a bye.

In how many ways can the first round of the competition be arranged if

(i) there are no restrictions? **2**

(ii) Amy is not playing Ben? **2**

End of Question 6

Marks

Question 7 (15 marks) Use a SEPARATE writing booklet.

(a) Suppose $x > 0, \quad y > 0, \quad z > 0$.

(i) Prove that $x^2 + y^2 \geq 2xy$. **1**

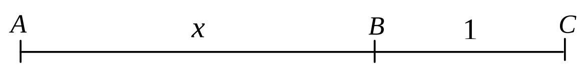
(ii) Hence, or otherwise, prove that $\frac{x}{y} + \frac{y}{z} \geq 2$. **1**

(iii) Prove that $x^3 + y^3 \geq xyz\left(\frac{x}{z} + \frac{y}{z}\right)$. **1**

(iv) Hence show that $x^3 + y^3 + z^3 \geq 3xyz$. **1**

(v) Deduce that $(a+b+c)(a+b+d)(a+c+d)(b+c+d) \geq 81abcd$ **1**
where $a > 0, \quad b > 0, \quad c > 0, \quad d > 0$.

(b) (i) **1**



The diagram shows a straight line segment AC divided by B in the ratio $x:1$.

If A divides CB externally in the same ratio that B divides AC internally, show that

$$x^2 = x + 1$$

(ii) A sequence $\{F_n\}$, the Fibonacci numbers, is defined by $F_1 = 1, \quad F_2 = 1$ **3**
and $F_{n+1} = F_n + F_{n-1}$ for $n \geq 2$.

The golden ratio φ, and its conjugate root θ, are the positive and negative solutions to the equation in part (i).

Prove by induction, that the closed form expression for the Fibonacci numbers is given by

$$F_n = \frac{\varphi^n - \theta^n}{\sqrt{5}}.$$

Question 7 continues on page 13

Marks

Question 7 (continued)

(c) A projectile is fired vertically upwards with initial speed u. It experiences air resistance proportional to its speed as well as gravitational acceleration g, so that in its upwards flight, the equation of motion is $\ddot{x} = -g - kv$, for some constant $k > 0$ and where v is the velocity of the projectile.

(i) Show that the time T taken to reach its maximum height is given by **3**

$$T = \frac{1}{k}\log_e\left(1 + \frac{ku}{g}\right).$$

(ii) By first writing $\ddot{x}$ as $v\frac{dv}{dx}$, show that the maximum height of the particle H is given by **3**

$$H = \frac{u - gT}{k}.$$

End of Question 7

Marks

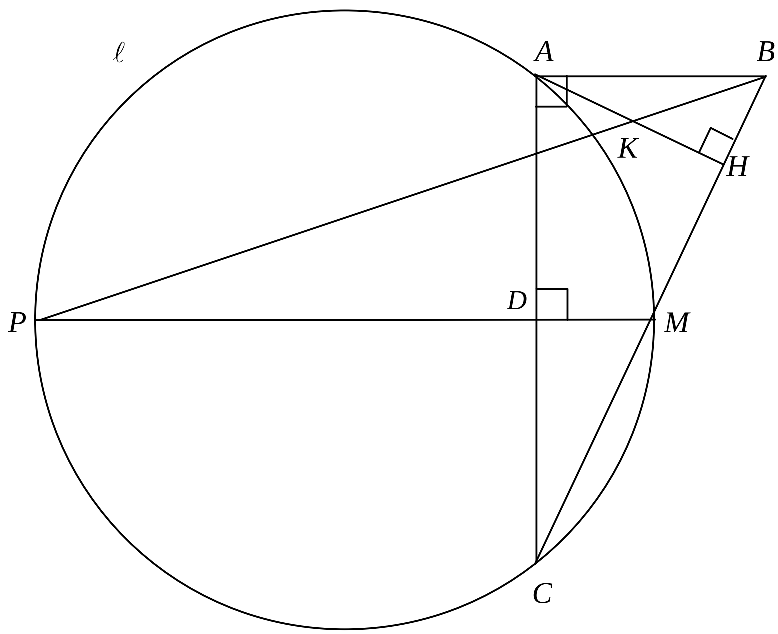
Question 8 (15 marks) Use a SEPARATE writing booklet.

(a) α is a double root of the equation $x^n - bx^2 + c = 0$.

(i) Show that $\alpha^2 = \dfrac{nc}{nb - 2b}$. **2**

(ii) Hence show that $n^n c^{n-2} = 4b^n (n-2)^{n-2}$. **2**

(b)



In ΔABC, $\angle A = 90°$, M is the midpoint of BC and H is the foot of the altitude from A to BC. A circle ℓ is drawn through points A, M and C. The line passing through M perpendicular to AC meets AC at D and the circle ℓ again at P. BP intersects AH at K.

(i) Show that PM is the diameter of the circle ℓ. **2**

(ii) Show that $\Delta MCD \parallel\!| \Delta MPC$. **2**

(iii) Hence show that $\Delta DMB \parallel\!| \Delta BMP$. **2**

(iv) Deduce that $\angle DBM = \angle ABK$. **2**

(v) By making further use of similar triangles, or otherwise, show that $AK = KH$. **3**

End of Paper

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STANDARD INTEGRALS

$$\int x^n \, dx = \frac{1}{n+1} x^{n+1}, \quad n \neq -1; \quad x \neq 0, \text{if } n < 0$$

$$\int \frac{1}{x} dx = \ln x, \quad x > 0$$

$$\int e^{ax} \, dx = \frac{1}{a} e^{ax}, \quad a \neq 0$$

$$\int \cos ax \, dx = \frac{1}{a} \sin ax, \quad a \neq 0$$

$$\int \sin ax \, dx = -\frac{1}{a} \cos ax, \quad a \neq 0$$

$$\int \sec^2 ax \, dx = \frac{1}{a} \tan ax, \quad a \neq 0$$

$$\int \sec ax \tan ax \, dx = \frac{1}{a} \sec ax, \quad a \neq 0$$

$$\int \frac{1}{a^2 + x^2} dx = \frac{1}{a} \tan^{-1} \frac{x}{a}, \quad a \neq 0$$

$$\int \frac{1}{\sqrt{a^2 - x^2}} dx = \sin^{-1} \frac{x}{a}, \quad a > 0, \quad -a < x < a$$

$$\int \frac{1}{\sqrt{x^2 - a^2}} dx = \ln\left(x + \sqrt{x^2 - a^2}\right), \quad x > a > 0$$

$$\int \frac{1}{\sqrt{x^2 + a^2}} dx = \ln\left(x + \sqrt{x^2 + a^2}\right)$$

NOTE: $\ln x = \log_e x, \quad x > 0$

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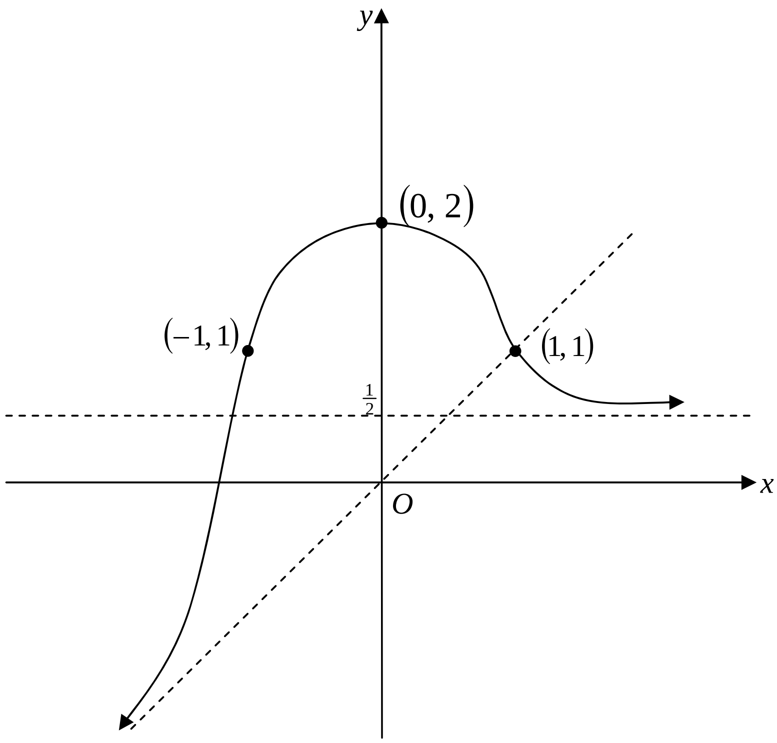
Mathematics Extension 2

Centre Number

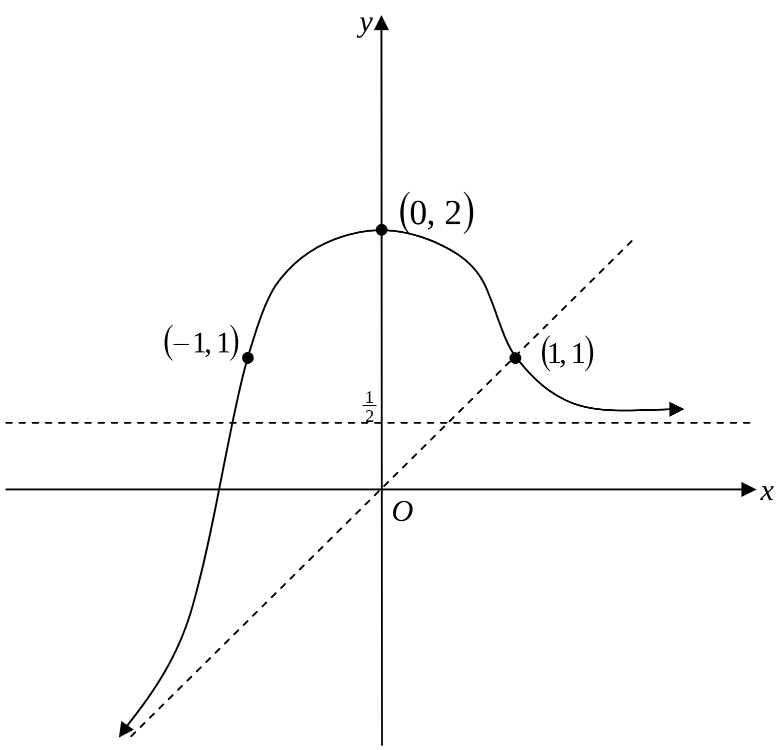
Questions 3 (b)

Student Number

(i)



(ii)



Question 3b (continued)

(iii)