SCEGGS Darlinghurst

Centre Number

Student Number

2003
HIGHER SCHOOL CERTIFICATE
TRIAL EXAMINATION

Mathematics Extension 2

This is a TRIAL PAPER only and does not necessarily reflect the content or format of the Higher School Certificate Examination for this subject.

General Instructions

- Reading time – 5 minutes
- Working time – 3 hours
- Write using black or blue pen
- Board-approved calculators may be used
- A table of standard integrals is provided at the back of this paper
- All necessary working should be shown in every question

Total marks - 120
- Attempt Questions 1–8
- All questions are of equal value

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Answer each question on a NEW page.

Marks

Question 1 (15 marks)

(a) Find

(i) $\displaystyle\int \frac{e^x}{(1+e^x)^2}\,dx$ — 1

(ii) $\displaystyle\int x\cos x\,dx$ — 2

(iii) $\displaystyle\int \frac{2x-3}{x^2-4x+8}\,dx$ — 3

(b) (i) Find real numbers A, B and C such that: — 2

$$\frac{10}{(3+x)(1+x^2)} \equiv \frac{A}{3+x} + \frac{Bx+C}{1+x^2} \quad (x \neq -3)$$

(ii) Hence find $\displaystyle\int \frac{10}{3+\tan\theta}\,d\theta$ using the substitution $x=\tan\theta$. — 3

(c) Use the substitution $x=\sin^2\theta$ to evaluate — 4

$$\int_0^{\frac{1}{2}} \frac{\sqrt{x}}{(1-x)^{\frac{3}{2}}}\,dx$$

Marks

Question 2 (15 marks) Start a NEW page.

(a) By considering the expansion of $(1+x)^{2n}$ in ascending powers of x, prove that **3**

$$\sum_{k=0}^{2n}\binom{2n}{k}=4^n$$

(b) Clearly indicate on an Argand Diagram the regions in the complex plane satisfied by:

(i) $0 \le \arg z \le \frac{\pi}{3}$ and $2 \le \operatorname{Im} z \le 3$. **2**

(ii) $|z-2i| \le 2$ and $|z-2-2i| \le 2$. **3**

(c) (i) Express $z=-1+i\sqrt{3}$ in modulus argument form. **1**

(ii) Hence or otherwise, confirm that z is a solution of the equation **3**

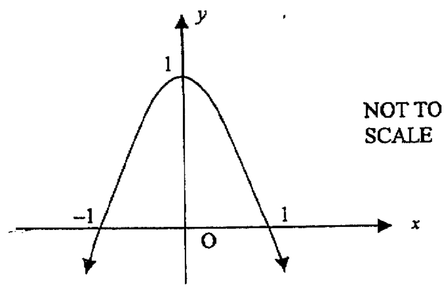
$$z^4-4z^2-16z-16=0$$

(iii) Find the other three solutions of the equation. **3**

Marks

Question 3 (15 marks) Start a NEW page.

(a)



The graph is of the curve $y=1-x^2$ where $f(x)=1-x^2$.

Without using calculus, sketch the following showing all important features.

(i) $y=\dfrac{-1}{f(x)}$ **1**

(ii) $|y|=|f(x)|$ **2**

(iii) $y=f(e^x)$ **2**

(iv) $y=\log_e(f(x))$ **2**

Question 3 continues on page 5

Question 3 (continued) — Marks

(b) Consider the cubic — 3

$$P(x) = x^3 + Ax + B \qquad \text{(where } A \text{ and } B \text{ are real)}$$

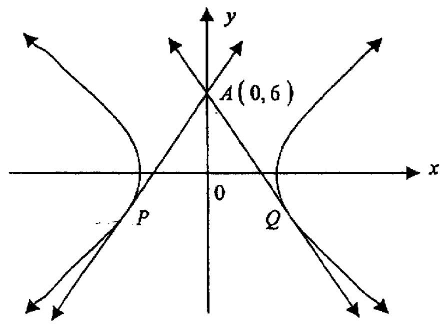
Prove that $P(x) = 0$ has exactly one real root if $A \geq 0$.

(c) (i) Find the points of intersection of the curve $y = x^2 - 2x$ and the straight line $y = x$. — 1

(ii) Use the method of cylindrical shells to find the volume generated when the region enclosed by the parabola $y = x^2 - 2x$ and the line $y = x$ is rotated about the y axis. — 4

Question 4 (15 marks) Start a NEW page. — Marks

(a)



NOT TO SCALE

The diagram shows the hyperbola $9x^2 - y^2 = 36$. Tangents from the point $A(0, 6)$ touch the hyperbola at the points $P(x_1, y_1)$ and $Q(x_2, y_2)$.

(i) Prove that the equation of the tangent at P is: — 2

$$y - y_1 = \frac{9x_1}{y_1}(x - x_1)$$

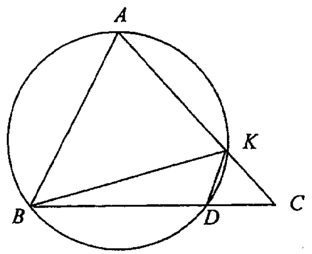
(ii) Prove that $9x_1^2 - y_1^2 = -6y_1$. — 2

(iii) Hence find the co-ordinates of P and Q. — 2

Question 4 continues on page 7

Question 4 (continued) **Marks**

(b)



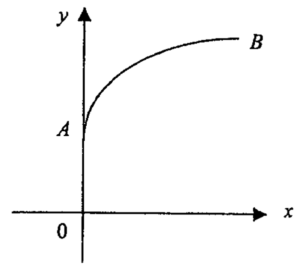
NOT TO SCALE

In the triangle ABC, $AB = AC$.
BK bisects $< ABC$.
Points A, B, D, K lie on the circumference of a circle.

(i) Assuming $< ABK = \alpha$, explain why $< DKC = 2\alpha$. **1**

(ii) Hence prove $AK = DC$. **3**

(c)



NOT TO SCALE

Points $A(0, 1)$ and $B(3, 2)$ lie on the curve $y^2 = x + 1$.
The region bounded by the curve, the line $y = 1$ and the line $x = 3$ is rotated about $x = 3$.

(i) By taking slices perpendicular to $x = 3$, prove that the volume formed is **3**

$$\pi \int_1^2 (4 - y^2)^2 \, dy$$

(ii) Hence find this volume. **2**

Question 5 (15 marks) Start a NEW page. **Marks**

(a) Without the use of calculus, draw a sketch of $y = \dfrac{\sin x}{x}$ for $-3\pi \le x \le 3\pi$. **3**

(b) $P(a\cos\theta, b\sin\theta)$ is any point on the ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$.

M is the midpoint of SP where S is a focus of the ellipse.

(i) Find the co-ordinates of M. **1**

(ii) Find the Cartesian equation of the locus of M. **2**

(iii) Prove that the locus is a second ellipse with centre at the midpoint of OS, where O is the origin. **2**

(c) (i) Explain the difficulty of using the formula for $\tan(A + B)$ when simplifying $\tan\left(A + \dfrac{\pi}{2}\right)$. **1**

(ii) Prove that $\tan\left(A + \dfrac{\pi}{2}\right) = -\cot A$. **2**

(iii) Hence use the method of Mathematical Induction to prove that **4**

$$\tan\left[(2n+1)\frac{\pi}{4}\right] = (-1)^n \text{ for all integers } n \ge 1.$$

Question 6 (12 marks) Start a NEW page. **Marks**

(a) Use the substitution $t = \tan\frac{\theta}{2}$ to find $\int \frac{d\theta}{1-\cos\theta-\sin\theta}$. **4**

(b) Given $I_n = \int_0^{\frac{\pi}{2}} \cos^{2n} x \, dx$,

(i) Prove that $I_n = \frac{2n-1}{2n} I_{n-1}$. **4**

(ii) Hence evaluate $\int_0^{\frac{\pi}{2}} \cos^6 x \, dx$. **3**

(c) (i) Prove that for $a > 0$ and $n \neq 0$, $\log_{a^n} x = \frac{1}{n}\log_a x$. **2**

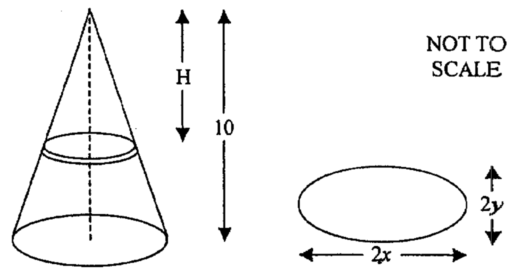
(ii) Hence evaluate in simplest form $\log_2 5 + \log_4 5 + \log_{16} 5 + \log_{256} 5 + \ldots$. **2**

Question 7 (15 marks) Start a NEW page. **Marks**

(a) (i) Evaluate $\int_{-a}^{a} \sqrt{a^2 - x^2} \, dx$ **1**

(ii) Explain how you could use the result in part (i) to prove that the area of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ is πab units^2. **2**

(iii) **4**



The ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ forms the base of a cone of height 10 units. A slice δH wide is taken H units from the vertex as shown. The cross section is an ellipse with major and minor axes $2x$ and $2y$ respectively.

Use the result from part (ii) to prove that the area of cross section H units from the vertex is $\frac{\pi ab H^2}{100}$ units^2

(iv) Hence find the volume of the right elliptical cone. **2**

Question 7 continues on page 11

Question 7 (continued) **Marks**

(b) (i) Expand $(a+b)^3$ **1**

(ii) Use this expansion and de Moivre's Theorem to prove that **2**

$$\cos 3\theta = 4\cos^3\theta - 3\cos\theta$$

(iii) If $\cos 3\theta = \frac{1}{2}$ and $x = \cos\theta$, prove that $8x^3 - 6x - 1 = 0$. **2**

(iv) Hence prove that $\cos\frac{\pi}{9}\ \cos\frac{5\pi}{9}\ \cos\frac{7\pi}{9} = \frac{1}{8}$. **1**

Question 8 (15 marks) Start a NEW page. **Marks**

(a) From a point on the ground an object of mass m is projected vertically upwards with an initial speed of u. Air resistance is mkv^2 and g is the acceleration due to gravity.

(i) Using a diagram or otherwise explain why $\ddot{x} = -g - kv^2$. **1**

(ii) Prove that the displacement x metres above ground level is given by **4**

$$x = \frac{1}{2k}\left[\log_e\left(\frac{g+ku^2}{g+kv^2}\right)\right]$$

(iii) If the object reaches a height of 40m above the ground prove that **3**

$$u^2 = \frac{g}{k}\left(e^{80k} - 1\right)$$

(b) The equation of a curve is $x^2y^2 - x^2 + y^2 = 0$

(i) Prove $y^2 = \frac{x^2}{x^2+1}$. **1**

(ii) Explain why $-1 < y < 1$. **1**

(iii) Prove that $\frac{dy}{dx} = \left(\frac{y}{x}\right)^3$. **3**

(iv) Sketch the curve. **2**

END OF PAPER