



SCEGGS Darlinghurst

2007

HIGHER SCHOOL CERTIFICATE
TRIAL EXAMINATION

Mathematics Extension 2

This is a TRIAL PAPER only and does not necessarily reflect the content or format of the Higher School Certificate Examination for this subject.

General Instructions

- Reading time – 5 minutes
- Working time – 3 hours
- Write using black or blue pen
- Board-approved calculators may be used
- A table of standard integrals is provided at the back of this paper
- All necessary working should be shown in every question

Total marks – 120

- Attempt Questions 1–8
- All questions are of equal value

Total marks – 120
Attempt Questions 1–8
All questions are of equal value

Answer each question in a SEPARATE writing booklet. Extra writing booklets are available.

Marks

Question 1 (15 marks)

(a) Evaluate $\int_0^4 \frac{dx}{\sqrt{x^2 + 9}}$ **1**

(b) Prove that $\int_0^1 \frac{dx}{4-x^2} = \frac{1}{4} \log_e 3$ **3**

(c) Find:

(i) $\int \sin^2 \theta \cos^3 \theta \, d\theta$ **1**

(ii) $\int x^2 \cos x \, dx$ **3**

(d) Use the substitution $t = \tan \frac{\theta}{2}$ to evaluate $\int_0^{\frac{\pi}{3}} \frac{d\theta}{1 - \sin \theta}$ giving your answer in exact form. **4**

(e) Find $\int \frac{2x}{\sqrt{2x-x^2}} \, dx$ **3**

Question 2 (15 marks) Use a SEPARATE writing booklet.

- (a) (i) Find the value of the product $(-1 + \sqrt{3}i)(1 + i)$ in both Cartesian and mod-arg forms. 2
- (ii) Hence (or otherwise) find the exact value of $\cos \frac{11\pi}{12}$. 1
- (b) The locus of the point $P(x, y)$ which moves in the complex plane is represented by the equation $|z - 3i| = 2$.
- (i) Sketch the locus on an Argand Diagram. 1
- (ii) Show that the minimum value of $\arg z$ is $\cos^{-1}\left(\frac{2}{3}\right)$. 2
- (iii) State the value of $|z|$ when P is at the position of minimum argument. 1
- (c) The equation $x^2 - (p + iq)x + 3i = 0$, where p and q are real, has roots α and β .
- (i) Write down expressions for the sum and product of these roots. 1
- (ii) If it is given that the sum of the squares of the roots is 8, find values for p and q . 3

Question 2 continues on page 4

Question 2 (continued)

(d) In an Argand Diagram, the vectors z , w and $z + w$ are represented by the points A , C and B respectively. O is the origin.

(i) Draw a diagram representing these points. 1

It is given that $|z - w| = |z + w|$.

(ii) What type of quadrilateral is $OABC$? Give a reason for your answer. 1

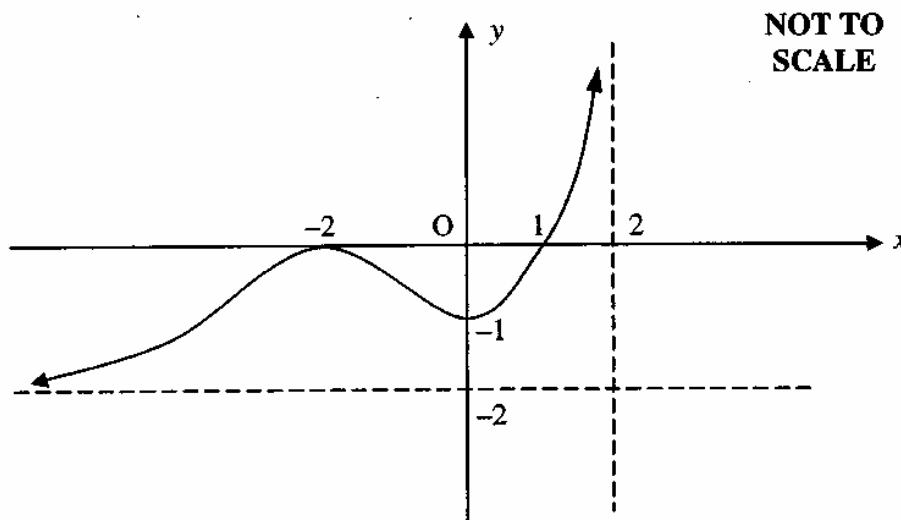
(iii) Find $\arg \frac{w}{z}$. 1

(iv) State a further condition necessary for $OABC$ to be a square. 1

End of Question 2

Question 3 (15 marks) Use a SEPARATE writing booklet.

- (a) The diagram shows the graph of $y = f(x)$.
 It has asymptotes $x = 2$ and $y = -2$.
 It has stationary points $(-2, 0)$ and $(0, -1)$.



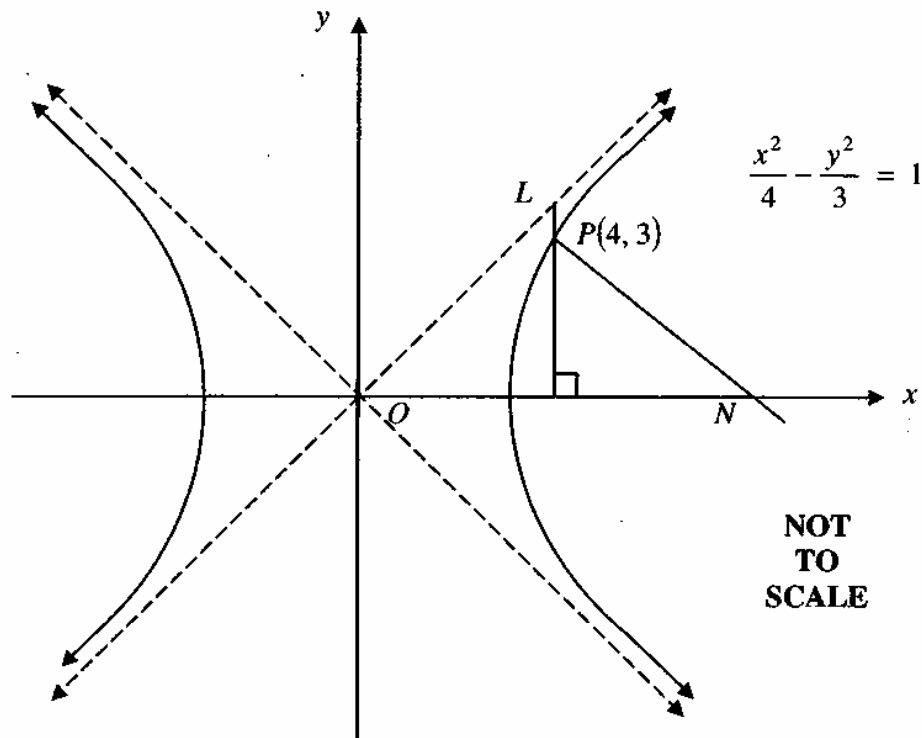
Draw separate one-third page sketches of the graphs of the following:
The above diagram has been reproduced on a page attached to the rear of this paper for you to use if you wish.

- | | | |
|-------|----------------|---|
| (i) | $y = f(x) $ | 2 |
| (ii) | $y = f(x - 2)$ | 2 |
| (iii) | $y^2 = -f(x)$ | 2 |
-
- (b) It is given that $x = 1 + i$ is a solution of $x^3 + x^2 - 4x + 6 = 0$.
- | | | |
|------|--|---|
| (i) | Find all the solutions of this equation. | 2 |
| (ii) | Hence solve $\frac{1}{2}x^2(x + 1) > 2x - 3$. | 2 |

Question 3 continues on page 6

Question 3 (continued)

- (c) The diagram shows the graph of the hyperbola $\frac{x^2}{4} - \frac{y^2}{3} = 1$.



The point $P(4, 3)$ lies on the hyperbola.

The normal at P to the hyperbola meets the x axis at N .

The vertical line through P meets the asymptote in the first quadrant at L .

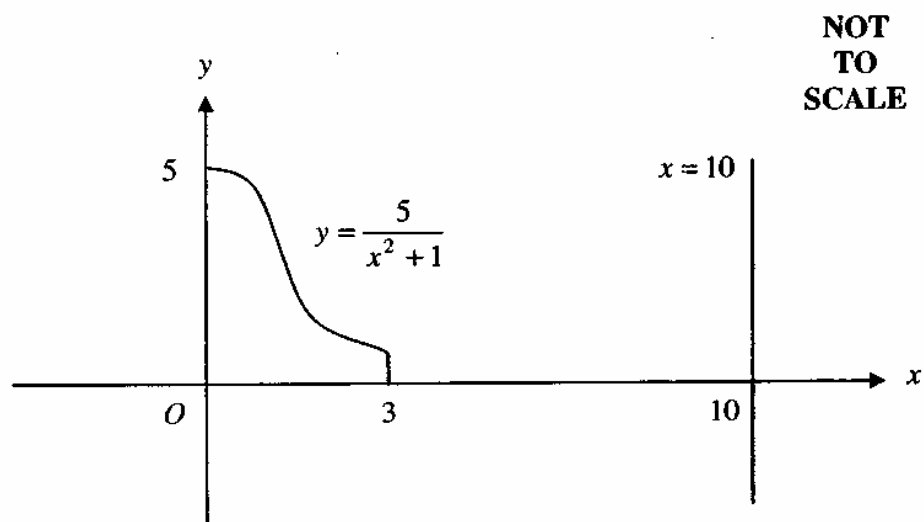
- (i) Show that the normal at P to the hyperbola has equation $x + y = 7$. 2
- (ii) Find $\angle OLN$. 3

End of Question 3

Question 4 (15 marks) Use a SEPARATE writing booklet.

- (a) The equation $x^4 + 4x^3 - 3x^2 - 4x - 2 = 0$ has roots α, β, γ and δ , 3
 Find the equation which has roots $\frac{2}{\alpha}, \frac{2}{\beta}, \frac{2}{\gamma}$ and $\frac{2}{\delta}$.

(b)



The region bounded by the curve $y = \frac{5}{x^2 + 1}$, the x -axis and the lines $x = 0$ and $x = 3$ is rotated about the line $x = 10$.

- (i) Use the method of cylindrical shells to show that the volume V is given by 3

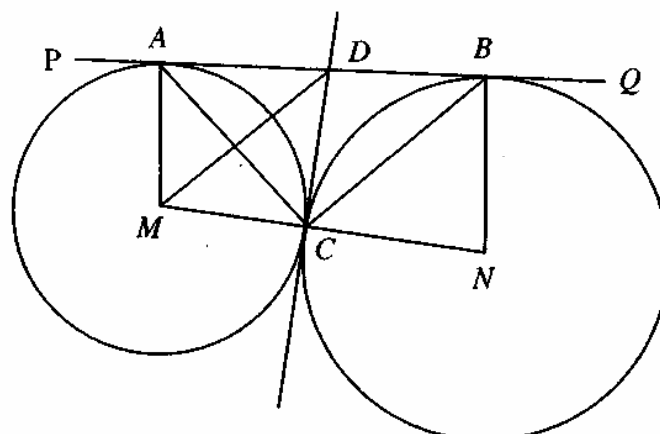
$$V = \int_0^3 \frac{100\pi - 10\pi x}{x^2 + 1} dx$$

- (ii) Hence find the volume V to 3 significant figures. 2

Question 4 continues page 8

Question 4 (continued)

(c)



NOT
TO
SCALE

M and N are the centres of two circles which touch at C . M , C and N are collinear.

PQ is a common tangent which touches the circles at A and B .

The tangent at C meets AB at D .

- (i) Explain why $ADCM$ and $DBNC$ are cyclic quadrilaterals. 2
- (ii) Prove that the triangles ACD and BNC are similar. 3
- (iii) Prove that MD is parallel to CB . 2

End of Question 4

Question 5 (15 marks) Use a SEPARATE writing booklet.

(a) It is given that

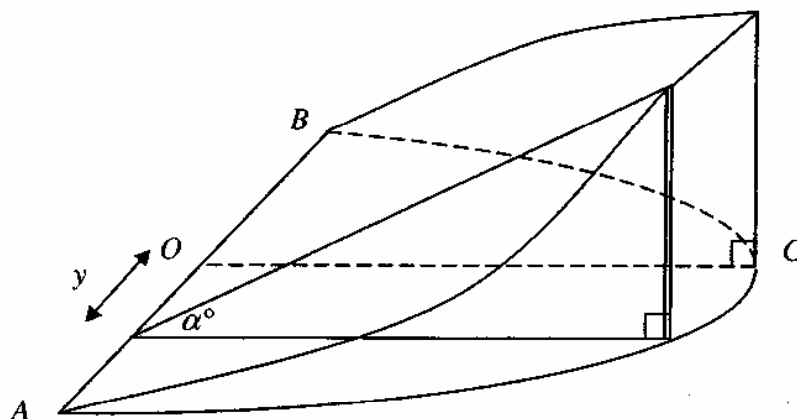
$$I_n = \int \tan^n x \, dx \text{ for } n \text{ a positive integer.}$$

(i) Prove that $I_n = \frac{1}{n-1} \tan^{n-1} x - I_{n-2}$ for $n \geq 2$. 3

(ii) Hence prove that 2

$$\int_0^{\frac{\pi}{4}} \tan^4 x \, dx = \frac{\pi}{4} - \frac{2}{3}$$

(b) The solid below is a wedge obtained by slicing a right elliptical cylinder (a cylinder whose base is an ellipse) at α° through its minor axis $AB = 2b$. The semi-major axis of the ellipse $OC = a$. A thin triangular slice of thickness δy perpendicular to the elliptical base and to the minor axis AB is drawn y units distant from OC .



NOT
TO
SCALE

(i) Prove that the volume of the triangular slice is 3

$$\delta V = \frac{a^2}{2b^2} \tan \alpha (b^2 - y^2) \delta y$$

(ii) Hence find the volume of the wedge. 2

Question 5 continues on page 10

Question 5 (continued)

Marks

- (c) (i) Find the equation of the normal to the rectangular hyperbola $xy = c^2$ at the point $\left(ct, \frac{c}{t}\right)$. **2**

- (ii) Show that if the normal crosses the y axis at the point $(0, h)$ then **1**

$$t^4 + \frac{h}{c}t - 1 = 0$$

- (iii) Hence prove that at most there are only two normals to the curve which can pass through $(0, h)$. **2**

End of Question 5

Question 6 (15 marks) Use a SEPARATE writing booklet.

(a) (i) Solve $\tan 3\theta = 1$ for $0 \leq \theta \leq \pi$. 1

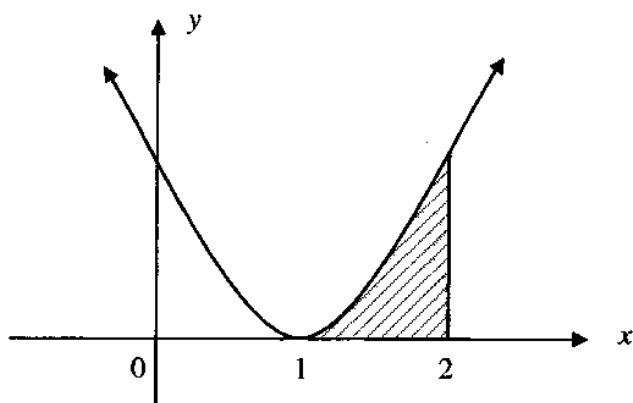
(ii) Given that $\sin 3\theta = 3\sin \theta \cos^2 \theta - \sin^3 \theta$ 1
 and $\cos 3\theta = \cos^3 \theta - 3\sin^2 \theta \cos \theta$,
 prove that $\tan 3\theta = \frac{3 \tan \theta - \tan^3 \theta}{1 - 3 \tan^2 \theta}$

(iii) Using the information in parts (i) and (ii) or otherwise, find θ where 2
 $x^3 - 3x^2 - 3x + 1 = 0$ and $x = \tan \theta$ if $0 \leq \theta \leq \pi$.

(iv) Hence prove that 2

$$\tan^2 \frac{\pi}{12} + \tan^2 5\frac{\pi}{12} = 14$$

(b) 4



NOT
TO
SCALE

The curve is $y = (x - 1)^2$.

The shaded region is bounded by the curve, the x axis and the line $x = 2$.

This region is rotated about the line $x = 2$.

Taking slices perpendicular to the y axis, find the volume formed.

Question 6 continues on page 10

Question 6 (continued)

(c) The tangent at the point $P(a^5 \cos^5 \theta, a^5 \sin^5 \theta)$ to the curve $x^{\frac{2}{5}} + y^{\frac{2}{5}} = a^2$, crosses the x axis at A and the y axis at B .

(i) Prove that A is the point $(a^5 \cos^3 \theta, 0)$ 3

(ii) Hence prove that the maximum area of the triangle AOB is: 2

$$\frac{a^{10}}{16} \text{ square units,}$$

where O is the origin.

End of Question 6

Question 7 (15 marks) Use a SEPARATE writing booklet.

- (a) A particle of unit mass is projected vertically upwards with velocity v against a constant gravitational acceleration g . Its equation of motion is given by $\dot{x} = -g - v$.

Initially, $x = 0$ and $v = k - g$ (where k is constant).

- (i) Prove that the time taken in upwards motion is given by: 3

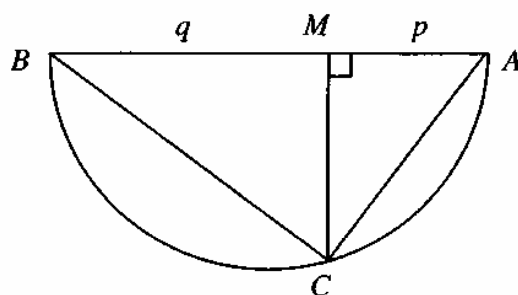
$$t = \log_e \frac{k}{v + g}$$

- (ii) Hence find the time taken for the particle to reach its highest point. 1

- (iii) Prove that the maximum height above the point of projection is: 4

$$x = k - g \left(1 + \log_e \frac{k}{g} \right).$$

- (b)



**NOT
TO
SCALE**

AB is the diameter of a circle, and C lies on the circumference of the circle.

MC is perpendicular to AB .

$AM = p$ units and $BM = q$ units.

- (i) Using similar triangles or otherwise prove that $MC = \sqrt{pq}$. 3

- (ii) Use the diagram to deduce that $\sqrt{pq} \leq \frac{p+q}{2}$. 2

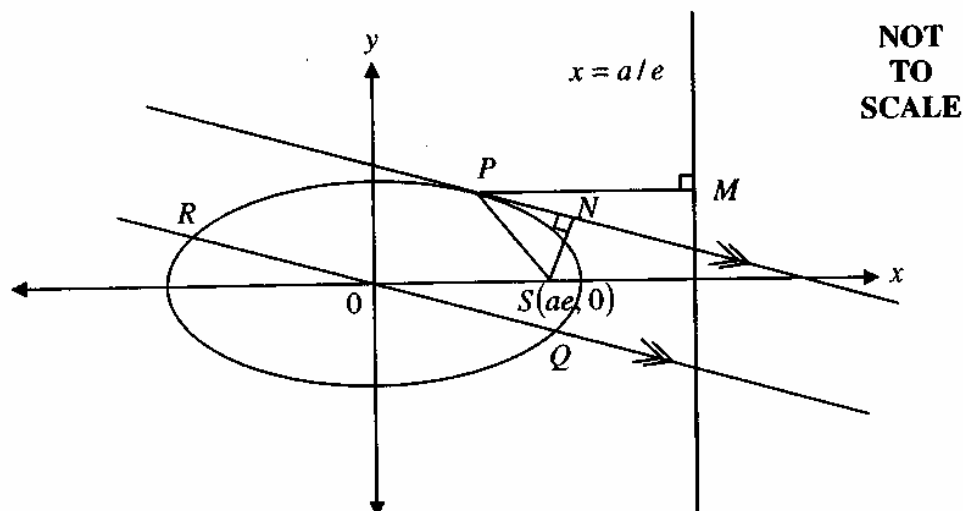
- (iii) Hence prove that if p, q, x and y are all positive. 2

$$\sqrt[4]{pqxy} \leq \frac{1}{4}(p + q + x + y)$$

Question 8 (15 marks) Use a SEPARATE writing booklet.

- (a) P is the point $(a \cos \theta, b \sin \theta)$ on the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.

The equation of the tangent at P is given as $bx \cos \theta + ay \sin \theta = ab$.



- (i) Prove that the equation of the diameter RQ which is parallel to the tangent at P is given by: 1

$$y = \frac{-b \cos \theta}{a \sin \theta} x$$

- (ii) Prove that the length of the diameter RQ is given by: 3

$$2\sqrt{b^2 \cos^2 \theta + a^2 \sin^2 \theta}$$

- (iii) Explain why $PS = a(1 - e \cos \theta)$ where e is the eccentricity of the ellipse. 1

- (iv) N is the foot of the perpendicular from the focus S to the tangent. Find SN . 1

- (v) Hence prove $b \times PS = OQ \times SN$. 2

Question 8 continues on page 15

Question 8 (continued)

Marks

- (b) (i) Show that $2\sin^2 x = 1 - \cos 2x$ and hence prove

2

$$\frac{\cos y - \cos(y + 2x)}{2\sin x} = \sin(y + x)$$

- (ii) Use Mathematical Induction to prove that

3

$$\sin x + \sin 3x + \sin 5x + \dots + \sin(2n - 1)x = \frac{1 - \cos 2nx}{2\sin x}$$

for n a positive integer.

- (iii) Hence solve

2

$$\sin x + \sin 3x + \sin 5x = 0$$

for $0 \leq x \leq \frac{\pi}{2}$

End of Paper

--	--	--	--	--

Centre Number

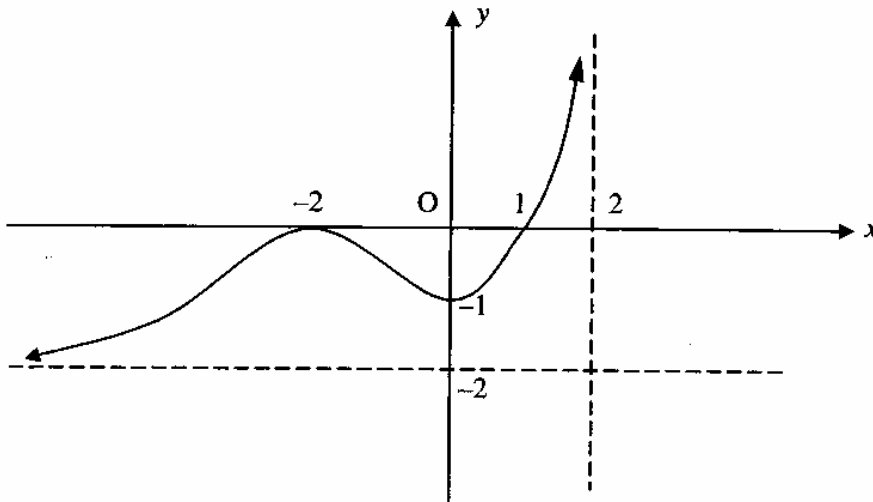
Questions 3 (a)

--	--	--	--	--	--	--	--	--	--

Student Number

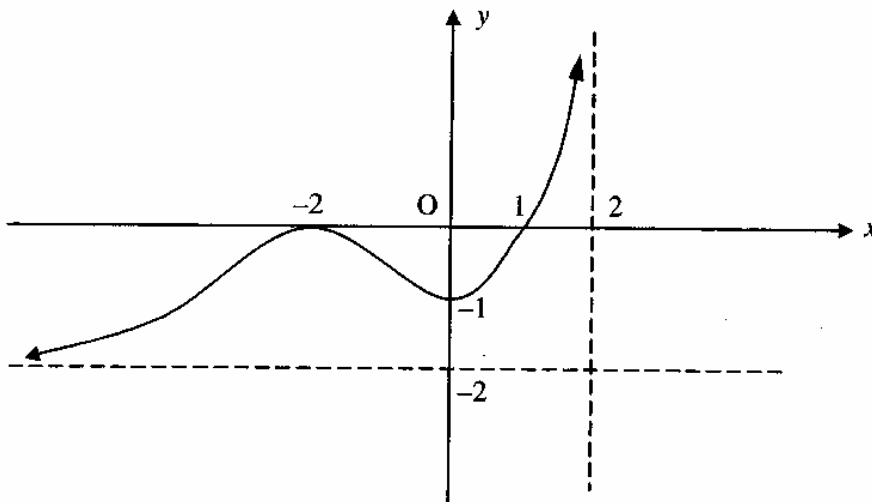
(i) $y = |f(x)|$

NOT TO
SCALE



(ii) $y = f(x - 2)$

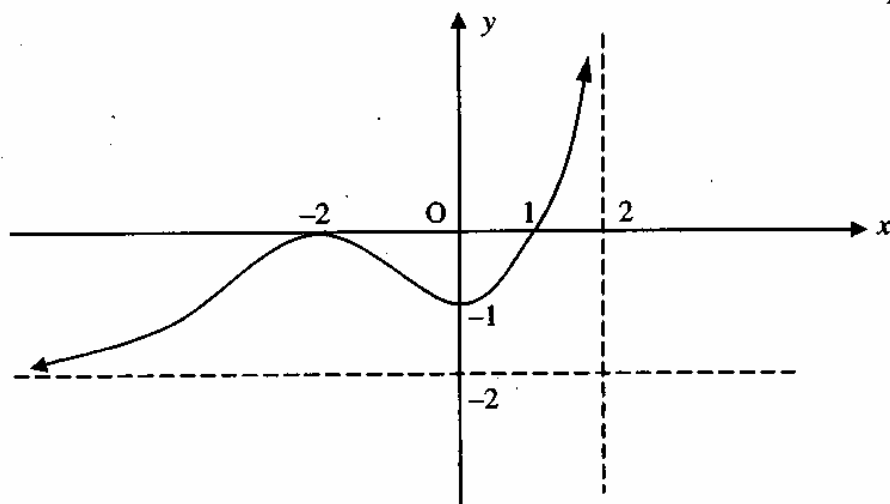
NOT TO
SCALE



Question 3 (a) continues overleaf

Question 3 (continued)

(a) (iii) $y^2 = -f(x)$



**NOT TO
SCALE**