SCEGGS Darlinghurst

2007

HIGHER SCHOOL CERTIFICATE
TRIAL EXAMINATION

Mathematics Extension 2

This is a TRIAL PAPER only and does not necessarily reflect the content or format of the Higher School Certificate Examination for this subject.

General Instructions

- Reading time – 5 minutes
- Working time – 3 hours
- Write using black or blue pen
- Board-approved calculators may be used
- A table of standard integrals is provided at the back of this paper
- All necessary working should be shown in every question

Total marks – 120

- Attempt Questions 1–8
- All questions are of equal value

Total marks – 120
Attempt Questions 1–8
All questions are of equal value

Answer each question in a SEPARATE writing booklet. Extra writing booklets are available.

Marks

Question 1 (15 marks)

(a) Evaluate $\displaystyle\int_0^4 \frac{dx}{\sqrt{x^2+9}}$ **1**

(b) Prove that $\displaystyle\int_0^1 \frac{dx}{4-x^2} = \frac{1}{4}\log_e 3$. **3**

(c) Find:

(i) $\displaystyle\int \sin^2\theta\,\cos^3\theta\; d\theta$ **1**

(ii) $\displaystyle\int x^2\cos x\; dx$ **3**

(d) Use the substitution $t=\tan\dfrac{\theta}{2}$ to evaluate $\displaystyle\int_0^{\frac{\pi}{3}} \frac{d\theta}{1-\sin\theta}$ giving your answer in exact form. **4**

(e) Find $\displaystyle\int \frac{2x}{\sqrt{2x-x^2}}\; dx$ **3**

Marks

Question 2 (15 marks) Use a SEPARATE writing booklet.

(a) (i) Find the value of the product $\left(-1+\sqrt{3}i\right)\left(1+i\right)$ in both Cartesian and mod-arg forms. **2**

(ii) Hence (or otherwise) find the exact value of $\cos\dfrac{11\pi}{12}$. **1**

(b) The locus of the point $P(x, y)$ which moves in the complex plane is represented by the equation $|z-3i| = 2$.

(i) Sketch the locus on an Argand Diagram. **1**

(ii) Show that the minimum value of arg z is $\cos^{-1}\left(\dfrac{2}{3}\right)$. **2**

(iii) State the value of $|z|$ when P is at the position of minimum argument. **1**

(c) The equation $x^2-(p+iq)x+3i=0$, where p and q are real, has roots α and β.

(i) Write down expressions for the sum and product of these roots. **1**

(ii) If it is given that the sum of the squares of the roots is 8, find values for p and q. **3**

Question 2 continues on page 4

Marks

Question 2 (continued)

(d) In an Argand Diagram, the vectors z, w and $z + w$ are represented by the points A, C and B respectively. O is the origin.

(i) Draw a diagram representing these points. **1**

It is given that $|z - w| = |z + w|$.

(ii) What type of quadrilateral is OABC? Give a reason for your answer. **1**

(iii) Find $\arg \dfrac{w}{z}$. **1**

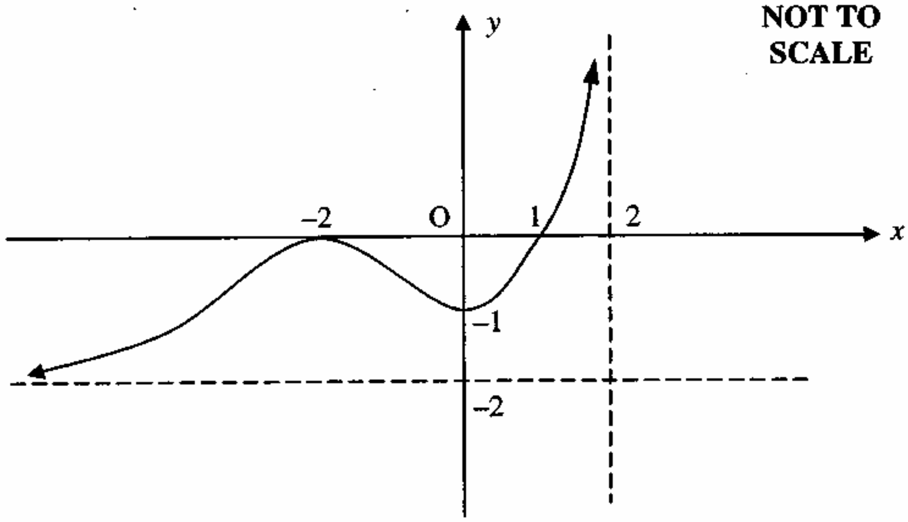
(iv) State a further condition necessary for OABC to be a square. **1**

End of Question 2

Marks

Question 3 (15 marks) Use a SEPARATE writing booklet.

(a) The diagram shows the graph of $y = f(x)$.
It has asymptotes $x = 2$ and $y = -2$.
It has stationary points $(-2, 0)$ and $(0, -1)$.



Draw separate one-third page sketches of the graphs of the following:
The above diagram has been reproduced on a page attached to the rear of this paper for you to use if you wish.

(i) $y = |f(x)|$ **2**

(ii) $y = f(x-2)$ **2**

(iii) $y^2 = -f(x)$ **2**

(b) It is given that $x = 1 + i$ is a solution of $x^3 + x^2 - 4x + 6 = 0$.

(i) Find all the solutions of this equation. **2**

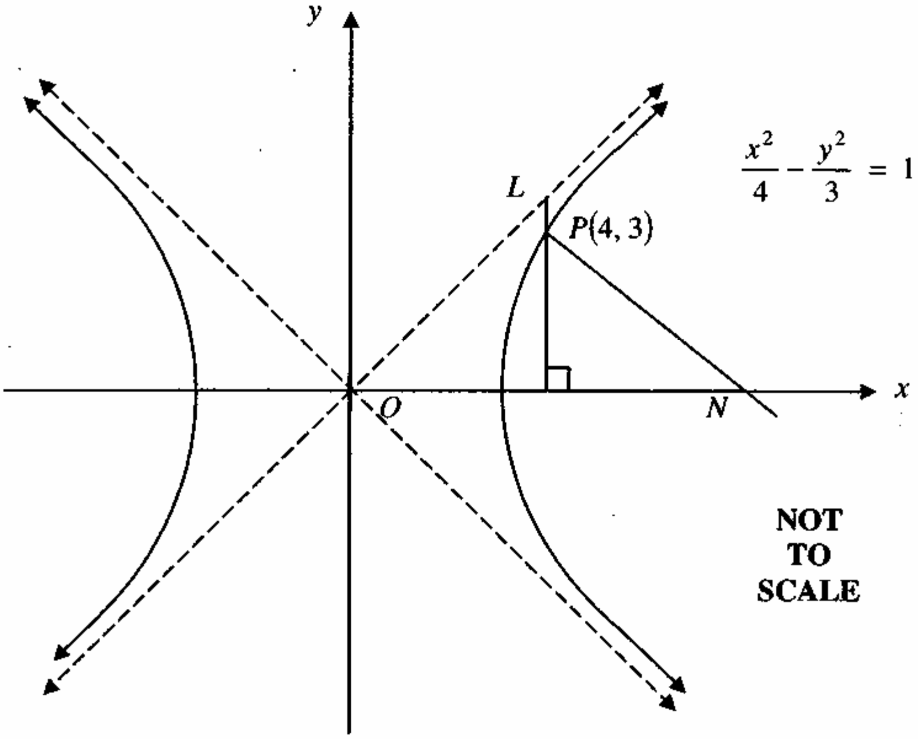
(ii) Hence solve $\frac{1}{2}x^2(x+1) > 2x - 3$. **2**

Question 3 continues on page 6

Marks

Question 3 (continued)

(c) The diagram shows the graph of the hyperbola $\frac{x^2}{4} - \frac{y^2}{3} = 1$.



The point $P(4, 3)$ lies on the hyperbola.
The normal at P to the hyperbola meets the x axis at N.
The vertical line through P meets the asymptote in the first quadrant at L.

(i) Show that the normal at P to the hyperbola has equation $x + y = 7$. **2**

(ii) Find $\angle OLN$. **3**

End of Question 3

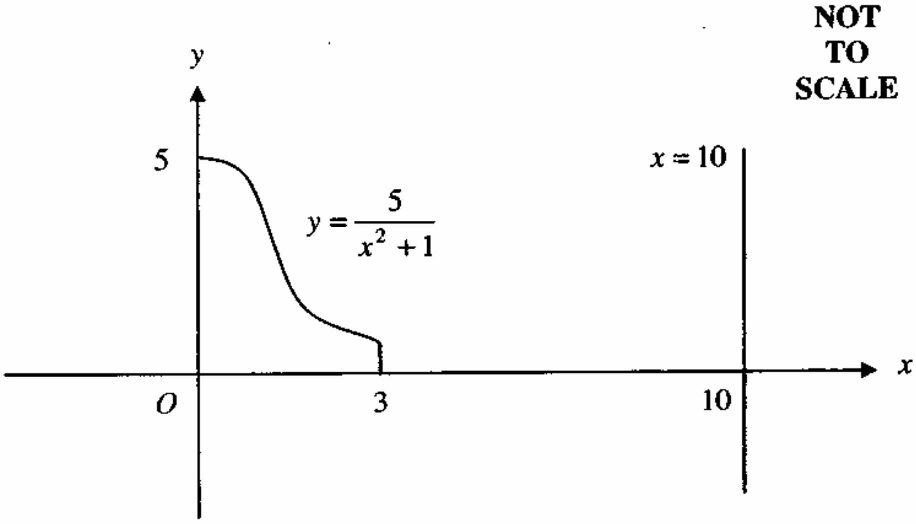
Marks

Question 4 (15 marks) Use a SEPARATE writing booklet.

(a) The equation $x^4 + 4x^3 - 3x^2 - 4x - 2 = 0$ has roots α, β, γ and δ, **3**

Find the equation which has roots $\frac{2}{\alpha}, \frac{2}{\beta}, \frac{2}{\gamma}$ and $\frac{2}{\delta}$.

(b)



The region bounded by the curve $y = \frac{5}{x^2+1}$, the x-axis and the lines $x = 0$ and $x = 3$ is rotated about the line $x = 10$.

(i) Use the method of cylindrical shells to show that the volume V is given by **3**

$$V = \int_0^3 \frac{100\pi - 10\pi x}{x^2+1}\, dx$$

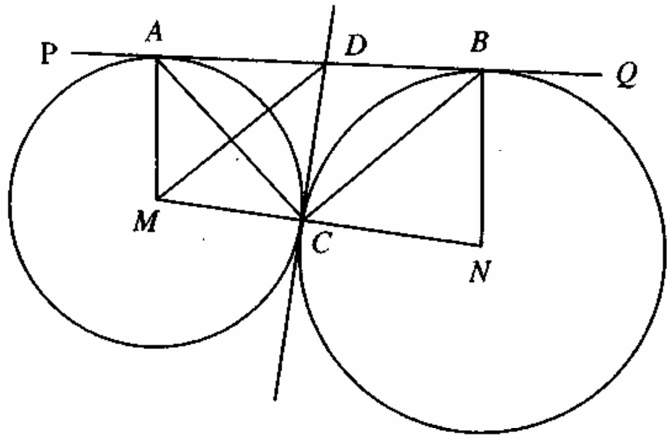
(ii) Hence find the volume V to 3 significant figures. **2**

Question 4 continues page 8

Marks

Question 4 (continued)

(c)



NOT TO SCALE

M and N are the centres of two circles which touch at C. M, C and N are collinear.

PQ is a common tangent which touches the circles at A and B.

The tangent at C meets AB at D.

(i) Explain why $ADCM$ and $DBNC$ are cyclic quadrilaterals. **2**

(ii) Prove that the triangles ACD and BNC are similar. **3**

(iii) Prove that MD is parallel to CB. **2**

End of Question 4

Marks

Question 5 (15 marks) Use a SEPARATE writing booklet.

(a) It is given that

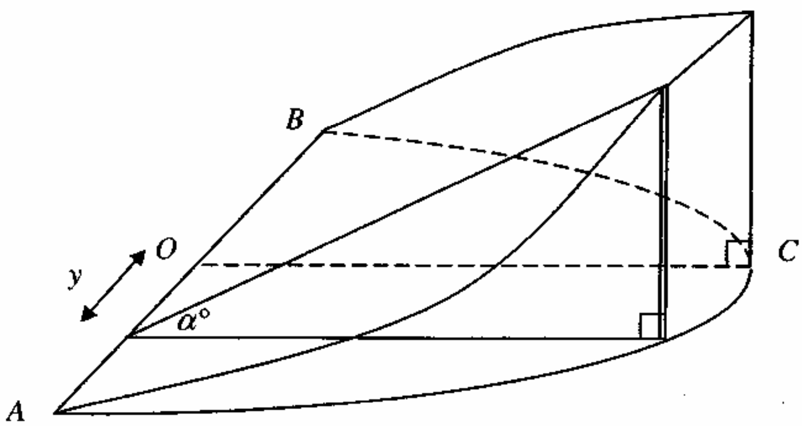
$$I_n = \int \tan^n x \, dx \quad \text{for } n \text{ a positive integer.}$$

(i) Prove that $I_n = \dfrac{1}{n-1}\tan^{n-1} x - I_{n-2}$ for $n \geq 2$. **3**

(ii) Hence prove that **2**

$$\int_0^{\frac{\pi}{4}} \tan^4 x \, dx = \frac{\pi}{4} - \frac{2}{3}$$

(b) The solid below is a wedge obtained by slicing a right elliptical cylinder (a cylinder whose base is an ellipse) at $\alpha°$ through its minor axis $AB = 2b$.
The semi-major axis of the ellipse $OC = a$.
A thin triangular slice of thickness δy perpendicular to the elliptical base and to the minor axis AB is drawn y units distant from OC.



NOT TO SCALE

(i) Prove that the volume of the triangular slice is **3**

$$\delta V = \frac{a^2}{2b^2}\tan\alpha\left(b^2 - y^2\right)\delta y$$

(ii) Hence find the volume of the wedge. **2**

Question 5 continues on page 10

Question 5 (continued) **Marks**

(c) (i) Find the equation of the normal to the rectangular hyperbola $xy = c^2$ at the point $\left(ct,\ \dfrac{c}{t}\right)$. **2**

(ii) Show that if the normal crosses the y axis at the point $(0, h)$ then **1**

$$t^4 + \frac{h}{c}t - 1 = 0$$

(iii) Hence prove that at most there are only two normals to the curve which can pass through $(0, h)$. **2**

End of Question 5

Marks

Question 6 (15 marks) Use a SEPARATE writing booklet.

(a) (i) Solve $\tan 3\theta = 1$ for $0 \le \theta \le \pi$. **1**

(ii) Given that $\sin 3\theta = 3\sin\theta\cos^2\theta - \sin^3\theta$ **1**

and $\cos 3\theta = \cos^3\theta - 3\sin^2\theta\cos\theta$,

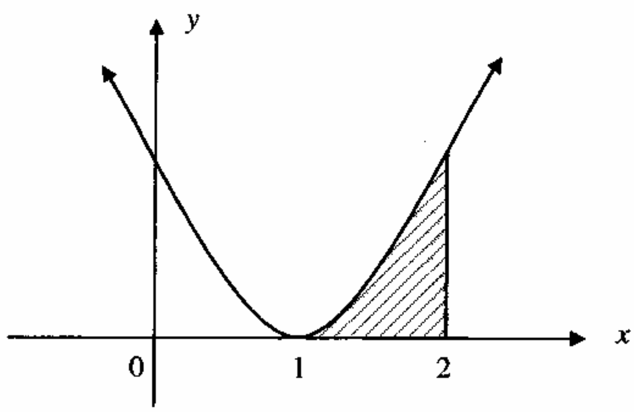
prove that $\tan 3\theta = \dfrac{3\tan\theta - \tan^3\theta}{1 - 3\tan^2\theta}$

(iii) Using the information in parts (i) and (ii) or otherwise, find θ where $x^3 - 3x^2 - 3x + 1 = 0$ and $x = \tan\theta$ if $0 \le \theta \le \pi$. **2**

(iv) Hence prove that **2**

$$\tan^2\frac{\pi}{12} + \tan^2 5\frac{\pi}{12} = 14$$

(b) **4**



NOT TO SCALE

The curve is $y = (x-1)^2$.

The shaded region is bounded by the curve, the x axis and the line $x = 2$.

This region is rotated about the line $x = 2$.

Taking slices perpendicular to the y axis, find the volume formed.

Question 6 continues on page 10

Marks

Question 6 (continued)

(c) The tangent at the point $P\left(a^5\cos^5\theta, a^5\sin^5\theta\right)$ to the curve $x^{\frac{2}{5}} + y^{\frac{2}{5}} = a^2$, crosses the x axis at A and the y axis at B.

(i) Prove that A is the point $\left(a^5\cos^3\theta, 0\right)$ **3**

(ii) Hence prove that the maximum area of the triangle AOB is: **2**

$$\frac{a^{10}}{16} \text{ square units,}$$

where O is the origin.

End of Question 6

Marks

Question 7 (15 marks) Use a SEPARATE writing booklet.

(a) A particle of unit mass is projected vertically upwards with velocity v against a constant gravitational acceleration g. Its equation of motion is given by $\ddot{x} = -g - v$.

Initially, $x = 0$ and $v = k - g$ (where k is constant).

(i) Prove that the time taken in upwards motion is given by: **3**

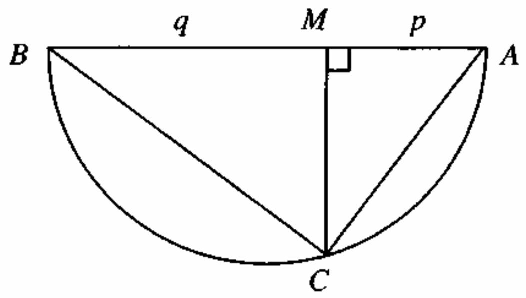
$$t = \log_e \frac{k}{v + g}$$

(ii) Hence find the time taken for the particle to reach its highest point. **1**

(iii) Prove that the maximum height above the point of projection is: **4**

$x = k - g\left(1 + \log_e \frac{k}{g}\right)$.

(b)



NOT TO SCALE

AB is the diameter of a circle, and C lies on the circumference of the circle.

MC is perpendicular to AB.

$AM = p$ units and $BM = q$ units.

(i) Using similar triangles or otherwise prove that $MC = \sqrt{pq}$. **3**

(ii) Use the diagram to deduce that $\sqrt{pq} \leq \frac{p + q}{2}$. **2**

(iii) Hence prove that if p, q, x and y are all positive. **2**

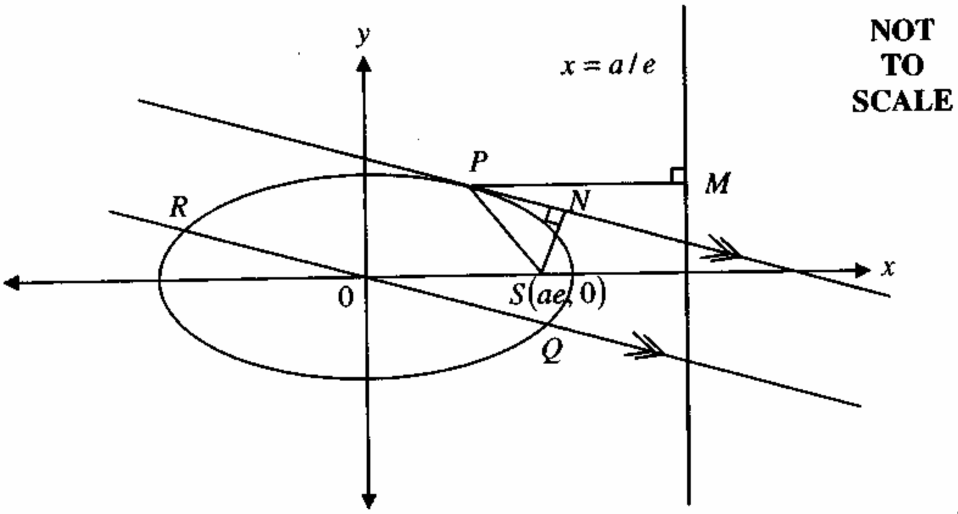
$$\sqrt[4]{pqxy} \leq \frac{1}{4}(p + q + x + y)$$

Marks

Question 8 (15 marks) Use a SEPARATE writing booklet.

(a) P is the point $(a\cos\theta, b\sin\theta)$ on the ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$.

The equation of the tangent at P is given as $bx\cos\theta + ay\sin\theta = ab$.



(i) Prove that the equation of the diameter RQ which is parallel to the tangent at P is given by: **1**

$$y = \frac{-b\cos\theta}{a\sin\theta}x$$

(ii) Prove that the length of the diameter RQ is given by: **3**

$$2\sqrt{b^2\cos^2\theta + a^2\sin^2\theta}$$

(iii) Explain why $PS = a(1 - e\cos\theta)$ where e is the eccentricity of the ellipse. **1**

(iv) N is the foot of the perpendicular from the focus S to the tangent. Find SN. **1**

(v) Hence prove $b \times PS = OQ \times SN$. **2**

Question 8 continues on page 15

Question 8 (continued) **Marks**

(b) (i) Show that $2\sin^2 x = 1 - \cos 2x$ and hence prove **2**

$$\frac{\cos y - \cos(y + 2x)}{2\sin x} = \sin(y + x)$$

(ii) Use Mathematical Induction to prove that **3**

$$\sin x + \sin 3x + \sin 5x + \ldots + \sin(2n-1)x = \frac{1 - \cos 2nx}{2\sin x}$$

for n a positive integer.

(iii) Hence solve **2**

$$\sin x + \sin 3x + \sin 5x = 0$$

for $0 \le x \le \frac{\pi}{2}$

End of Paper

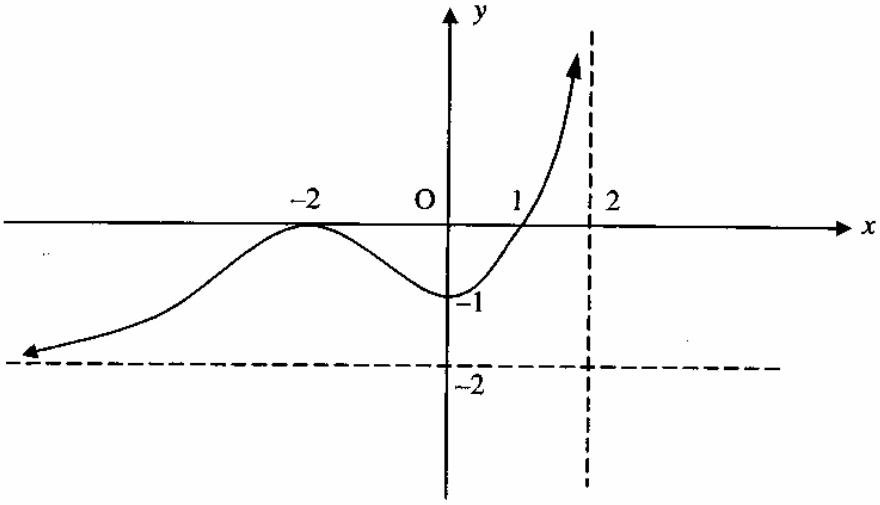
Centre Number

Student Number

Questions 3 (a)

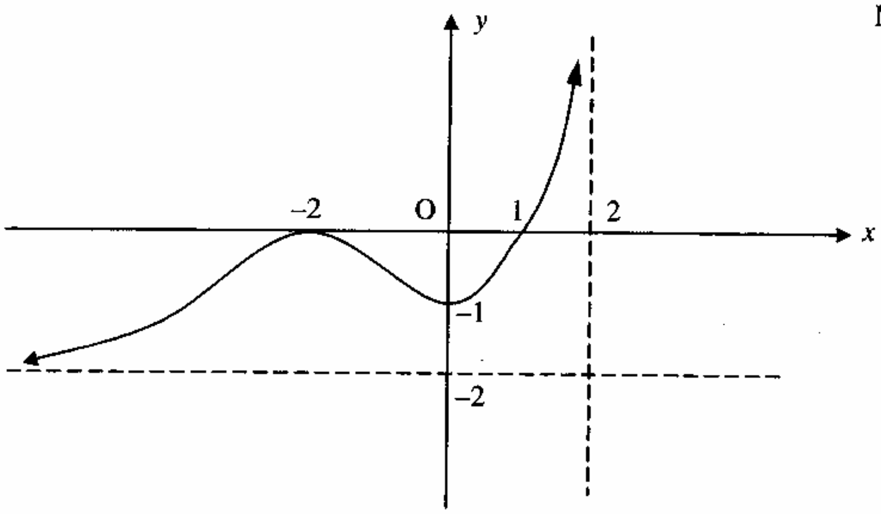
(i) $y = |f(x)|$

NOT TO SCALE



(ii) $y = f(x-2)$

NOT TO SCALE



Question 3 (a) continues overleaf

Question 3 (continued)

(a) (iii) $y^2 = -f(x)$

NOT TO SCALE