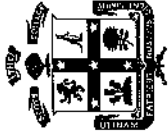


THE SCOTS COLLEGE



TRIAL HIGHER SCHOOL CERTIFICATE EXAMINATION

YEAR 12

EXTENSION 2 MATHEMATICS

AUGUST 2001

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Exam continues over

TIME ALLOWED: THREE HOURS
(PLUS 5 MINUTES READING TIME)

OUTCOMES:

- Uses the relationship between algebraic and geometric representations of complex numbers and of conic sections. [E3]
- Uses efficient techniques for algebraic manipulation required in dealing with questions such as those involving conic sections and polynomials. [E4]
- Combines the ideas of algebra and calculus to determine the important features of the graphs of a wide variety of functions. [E6]
- Uses the techniques of slicing and cylindrical shells to determine volumes. Applies further techniques of integration, including partial fractions, integration by parts and recurrence formulae, to problems. [E7]
- Applies further techniques of integration, including partial fractions, integration by parts and recurrence formulae, to problems. [E8]
- Uses ideas and techniques from calculus to solve problems in mechanics involving resolution of forces, resisted motion and circular motion. [E5]

[4 MARKS]

a. Let $z = 3 - 2i$ and $w = -5 + 6i$ (i) Find $\operatorname{Im}(uz)$ (ii) Find $|w - z|$ (iii) Find $\overline{-2iz}$ (iv) Express $\frac{w}{z}$ in the form $a + ib$, where a and b are real numbers.

[4 MARKS]

a. Find:

(i) $\int x^3 \log_e x \, dx$

(ii) $\int \sin^3 \theta \, d\theta$

b. Find the exact value of:

$$\int_1^e \frac{dx}{x^2 - 8x + 25}$$

[3 MARKS]

c. Using the substitution $u = \cos x$ to evaluate:

$$\int_0^{\frac{\pi}{2}} \frac{\sin^3 x}{\cos^2 x} \, dx$$

[3 MARKS]

d.

(i) Show that $(1 - \sqrt{x})^{n-1} \sqrt{x} = (1 - \sqrt{x})^{n-1} - (1 - \sqrt{x})^n$

(ii) If $I_n = \int_0^1 (1 - \sqrt{x})^n \, dx$ for $n \geq 0$ show that $I_n = \frac{n}{n+2} I_{n-1}$ for $n \geq 1$

(iii) Deduce that $\frac{1}{I_n} = \binom{n+2}{n}$ for $n \geq 0$

[5 MARKS]

[4 MARKS]

b. On separate Argand diagrams sketch:

(i) $\{z : |z - 2i| < 2\}$

(ii) $\{z : \arg(z - (1 + i)) = -\frac{3\pi}{4}\}$

[7 MARKS]

c. z_1 and z_2 are two complex numbers such that $\frac{z_1 + z_2}{z_1 - z_2} = 2i$ (i) On an Argand diagram show vectors representing: $z_1, z_2, z_1 + z_2$ and $z_1 - z_2$.(ii) Show that $|z_1| = |z_2|$ (iii) If α is the angle between the vectors representing z_1 and z_2 , show that $\tan \frac{\alpha}{2} = \frac{1}{2}$ (vi) Show that $z_2 = \frac{1}{5}(3 + 4i)z_1$

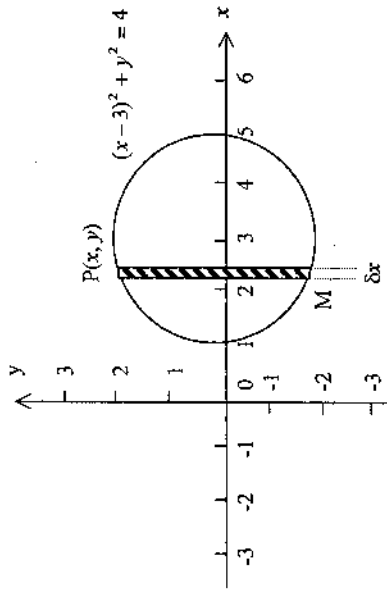
a. The base of a solid is the region between the lines $y = 3x$ and $y = -x$ from $x = 0$ to $x = 2$. Each cross section by planes perpendicular to the x axis is a square with its side determined by the base. Calculate the volume of the solid. [3 MARKS]

b. The area bounded by the curve $y = x^2 + 1$ and the line $y = 3 - x$ is rotated about the x -axis. [4 MARKS]

- (i) Sketch the curve and the line clearly showing and labelling all the points of intersection.
- (ii) By considering slices perpendicular to the x -axis, find the volume of the solid formed.

c. The graph below is of the circle $(x - 3)^2 + y^2 = 4$. [8 MARKS]

$P(x, y)$ is a point on the circumference of the circle. PM is the left-hand end of a strip of width δx which is parallel to the y -axis.



(i) Show, using the method of cylindrical shells, that the volume V of the doughnut-shaped solid formed when the region inside the circle is rotated about the y -axis is given by:

$$V = 4\pi \int_1^3 x\sqrt{4 - (x - 3)^2} dx$$

(ii) Hence, by using the substitution $u = x - 3$ or otherwise find the volume of the doughnut.

Consider the function $f(x) = x - 2\sqrt{x}$ [15 MARKS]

- a. Determine the domain of $f(x)$.
- b. Find the x intercepts of the graph of $y = f(x)$.
- c. Show that the curve $y = f(x)$ is concave upwards for all positive values of x .
- d. Find the coordinates of the turning point and determine its nature.
- e. Sketch the graph of $y = f(x)$ clearly showing all essential details.

f. Hence, sketch on separate diagrams:

- (i) $y = |f(x)|$
- (ii) $y = f(x - 1)$
- (iii) $y = f(|x|)$
- (iv) $|y| = f(x)$
- (v) $y = \frac{1}{f(x)}$

a. Given that $z = -1 + \sqrt{3}i$ is a root of the equation $z^4 - 4z^2 - 16z - 16 = 0$, find the other roots. [4 MARKS]

b. Given that α, β and γ are the roots of the cubic equation $x^3 - x^2 + 5x - 3 = 0$, find: [5 MARKS]

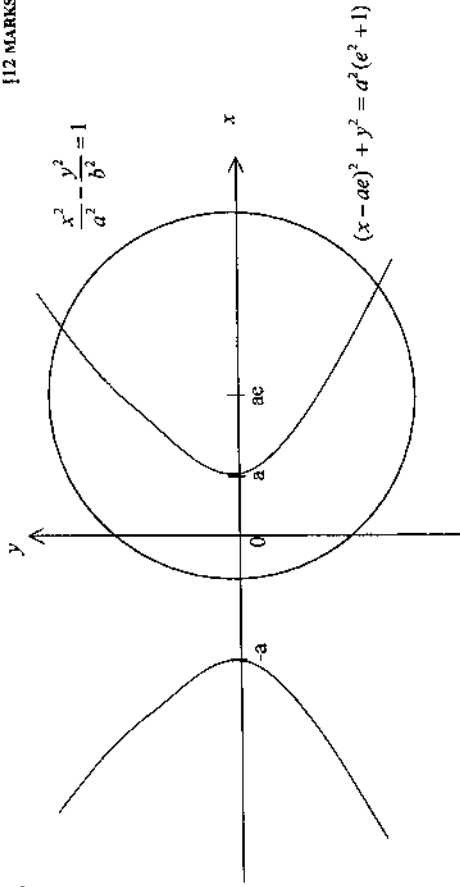
(i) the equation whose roots are $-\alpha, -\beta, -\gamma$.

(ii) the equation whose roots are $\alpha\beta, \alpha\gamma, \beta\gamma$.

c. For what values of m does the equation $x^3 - 12x^2 + 45x - m = 0$ have three distinct solutions? [6 MARKS]

a. A hyperbola has asymptotes $y = x$ and $y = -x$. It passes through the point $(3, 2)$. Find the equation of the hyperbola and determine its eccentricity and foci. [3 MARKS]

b. [12 MARKS]



(i) Show that the tangent at $P(a \sec \theta, b \tan \theta)$ on the hyperbola

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

has equation

$$\frac{x \sec \theta}{a} - \frac{y \tan \theta}{b} - 1 = 0$$

(ii) Show that if the tangent at P is also tangent to the circle with centre $(ae, 0)$ and radius $a\sqrt{e^2 + 1}$, then show $\sec \theta = -e$.

(iii) Given that $\sec \theta = -e$, deduce that the points of contact P, Q on the hyperbola of the common tangents to the circle and hyperbola are the extremities of a latus rectum of the hyperbola, and state the coordinates of P and Q .

(iv) Find the equations of the common tangents to the circle and hyperbola, and find the coordinates of their points of contact with the circle.

a. A mass of 10kg falls freely from rest through 10 metres and then comes to rest again after penetrating 0.2 metres of sand. Find the resistance of the sand, assumed constant. [4 MARKS]

b. A particle moving in a straight line experiences a force numerically equal to $\left(x + \frac{1}{x}\right)$ newtons per unit mass, towards the origin. The particle starts from rest, d units from the origin. [4 MARKS]

(i) Find an expression for its speed in terms of x .

(ii) Hence or otherwise, deduce its speed when it is half way from the origin.

c. An object of irregular shape and of mass 100kg is found to experience a resistive force, in newtons, of magnitude one-tenth the square of its velocity in metres per second when it moves through air [use $g = 9.8 \text{ms}^{-2}$]. [7 MARKS]

If the object falls from rest under gravity:

(i) show that acceleration is given by $a = g - \frac{v^2}{1000}$.

(ii) calculate its terminal velocity.

(iii) calculate the maximum height, to the nearest metre, of the release point above the ground, if the object attains a speed of 80% of its terminal velocity before striking the ground.

a. Let α , β and γ be the roots of the cubic equation $x^3 + Ax^2 + Bx + 8 = 0$, where A , and B are real. Furthermore $\alpha^2 + \beta^2 = 0$ and $\beta^2 + \gamma^2 = 0$. [5 MARKS]

(i) Explain why β is real and α and γ are not real.

(ii) Show that α and γ are purely imaginary.

(iii) Find A and B .

b. It is given that if $J_n = \int \cos^{n-1} x \sin nx \, dx$ and $n \geq 1$ then: [5 MARKS]

$$J_n = \frac{1}{2n-1} [(n-1)J_{n-1} - \cos^{n-1} x \cos nx]$$

Use this reduction formula to show that:

$$\int_0^{\frac{\pi}{2}} \cos^2 x \sin 3x \, dx = \frac{1}{60} (28 - \sqrt{2})$$

c. (i) Prove that $(1 + i \tan \theta)^n + (1 - i \tan \theta)^n = 2 \sec^n \theta \cos n\theta$ [5 MARKS]

(ii) Hence prove that $\text{Re}(1 + i \tan \frac{\pi}{8})^8 = 64(17 - 12\sqrt{2})$.