

**Question 1.** (15 marks)

a) Evaluate

i)  $\int \frac{dx}{(x+1)(x+3)}$

ii)  $\int \sqrt{4-x^2} dx$

iii)  $\int x\sqrt{2-x} dx$

b) Find  $\int x^2 e^{-x} dx$

c) If  $I_n = \int_0^{\frac{\pi}{2}} \sin^n x dx$ , show that  $I_n = \frac{n-1}{n} I_{n-2}$ .

Hence evaluate  $\int_0^{\frac{\pi}{2}} \sin^5 x dx$

**Question 2.** (15 marks)

a) Reduce the polynomial  $P(x) = x^4 - 2x^2 - 15$  into irreducible factors over  
 i) the rational field  $\mathbb{Q}$ ;  
 ii) the real field  $\mathbb{R}$ ;  
 iii) the complex field  $\mathbb{C}$ .

b) Divide the polynomial  $x^3 + 5x^2 - 7x - 3$  by  $(x - 2i)$  using long division.

c) Show that  $2 - \sqrt{3}$  is a zero of the polynomial  $a(x) = x^3 - 15x + 4$ .  
 Hence reduce  $a(x)$  to irreducible factors over the real field.

d) Given that the polynomial  $P(x) = x^4 + x^2 + 6x + 4$  has a rational zero of multiplicity 2, find all the zeros of  $P(x)$  over the complex field.

e) If  $\alpha, \beta, \gamma$  are the roots of the equation

$$x^3 + qx + r = 0,$$

where  $r \neq 0$ , obtain as functions of  $q$  and  $r$ , in their simplest forms, the coefficients of the cubic equation whose roots are  $\frac{1}{\alpha^2}, \frac{1}{\beta^2}, \frac{1}{\gamma^2}$ .

**Question 3.** (15 marks)

a) i) Define the modulus  $|z|$  of a complex number  $z$ .

ii) Given two complex numbers  $z_1, z_2$  prove that  $|z_1 z_2| = |z_1| |z_2|$ ;

b) Given

$$w = \frac{2-3i}{1-i}$$

determine

- i)  $|w|$  (the modulus of  $w$ );
- ii)  $\bar{w}$  (the conjugate of  $w$ );
- iii)  $w + \bar{w}$ .

c) Describe, in geometric terms, the locus (in the Argand plane) represented by  $2|z| = z + \bar{z} + 4$ .

**Question 4.** (15 marks)

a) Determine the real values of  $k$  for which the equation

$$\frac{x^2}{19-k} + \frac{y^2}{7-k} = 1$$

defines respectively an ellipse and a hyperbola.

Sketch the curve corresponding to the value  $k = 3$ .

Describe how the shape of this curve changes as  $k$  increases from 3 towards 7. What is the limiting position of the curve as 7 is approached?

b)  $P$  is a point on the ellipse  $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$  with centre  $O$ . A line drawn through  $O$ , parallel to the tangent to the ellipse at  $P$ , meets the ellipse at  $Q$  and  $R$ .

Prove that the area of triangle  $PQR$  is independent of the position of  $P$ .

**Question 5.** (15 marks)

a) Sketch the curve  $y^2 = x^2(x-2)(x-3)$ .

b) In the Cartesian plane sketch the curve

$$y = \frac{e^x - e^{-x}}{e^x + e^{-x}}$$

and prove that the lines  $y = \pm 1$  are asymptotes.

Also, if  $k$  is a positive constant, find the area in the positive quadrant enclosed by the above curve and the three lines

$$y = 1, x = 0, x = k$$

and prove that this area is always less than  $\ln 2$ , however large  $k$  may be.

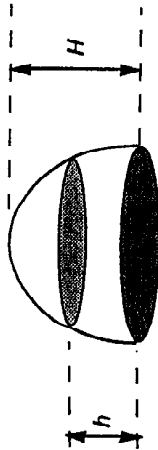
**Question 6.** (15 marks)

- a) The area bounded by the curve  $y = \frac{1}{x+1}$ , the x-axis, the line  $x=2$  and the line  $x=8$ , is rotated about the y-axis. Find the volume of the solid generated using the method of cylindrical shells.
- b) i) Using the substitution  $x = a \sin \theta$ , or otherwise, verify that

$$\int_a^b \sqrt{a^2 - x^2} dx = \frac{1}{2} \pi a^2.$$

- ii) Deduce that the area enclosed by the ellipse  $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$  is  $\pi ab$ .

iii)



The diagram shows a mound of height  $H$ . At height  $h$  above the horizontal base, the horizontal cross-section of the mound is elliptical in shape, with equation

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = \lambda^2,$$

where

$$\lambda = 1 - \frac{h^2}{H^2},$$

and  $x, y$  are appropriate coordinates in the plane of cross-section.

Show that the volume of the mound is  $\frac{8\pi abH}{15}$ .

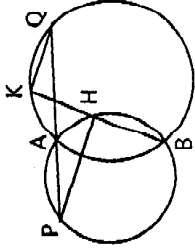
**Question 7.** (15 marks)

- a) Six letters are chosen from the letters of the word AUSTRALIA. These six letters are then placed alongside one another to form a six letter arrangement. Find the number of distinct six letter arrangements which are possible, considering all the choices.
- b) Solve for  $x$  the following inequation

$$\frac{x^2 - 5x}{4 - x} \leq -3$$

and show the solutions on a number line.

c)



In the figure  $PAQ$  and  $BHK$  are straight lines. Prove that  $PH$  is parallel to  $KQ$ .

- d) Two circles, centres  $B$  and  $C$ , touch externally at  $A$ .  $PQ$  is a direct common tangent touching the circles at  $P$  and  $Q$  respectively.
- i) Draw a neat diagram depicting the given information;
- ii) Prove that the circle with  $BC$  as diameter touches the line  $PQ$ .

**Question 8.** (15 marks)

- a) An aeroplane flies horizontally due East at a constant speed of 240 km/h. From a point  $P$  on the ground the bearing of the plane at one instant is  $311^\circ T$  and 3 minutes later the bearing of the plane is  $073^\circ T$  whilst its elevation then is  $21^\circ$ . If  $h$  metres is the altitude of the plane, show that

$$h = 12000 \sin 41^\circ \tan 21^\circ \operatorname{cosec} 58^\circ$$

and calculate  $h$  correct to the nearest metre.

- b) The magnitude and direction of the acceleration due to gravity at a point outside the Earth at a distance  $x$  from the Earth's centre is equal to  $-\frac{k}{x^2}$ , where  $k$  is a constant.

- i) Neglecting atmospheric resistance, prove that if an object is projected upwards from the Earth's surface with speed  $u$ , its speed  $v$  in any position is given by

$$v^2 = u^2 - 2gR^2 \left( \frac{1}{R} - \frac{1}{x} \right)$$

where  $R$  is the Earth's radius and  $g$  is the magnitude of the acceleration due to gravity at the Earth's surface.

- ii) Show that the greatest height,  $H$ , above the Earth's surface reached by the particle is given by

$$H = \frac{u^2 R}{2gR - u^2}.$$

- iii) Hence, if the radius of the Earth is approximately 6400 km, and the acceleration due to gravity at the Earth's surface is  $9.8 \text{ m/s}^2$ , find the speed required by the particle to escape the Earth's gravitational influence.

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