

Question 1 (15 marks) The Scots College Ex 2 Maths
 Nov 2003

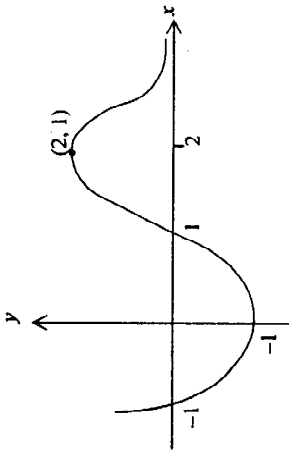
Marks

- | | | |
|---|---|---|
| a | Find $\int \frac{\ln x}{x} dx$ | 1 |
| b | By completing the square find $\int \frac{dx}{x^2 + 4x + 8}$ | 2 |
| c | Use integration by parts to find $\int \sin^{-1} x dx$ | 2 |
| d | Find $\int \frac{x^2}{x^2 - 9} dx$ | 3 |
| e | Evaluate $\int_0^{\frac{\pi}{4}} \sin 5x \cos 4x dx$ | 3 |
| f | Use the substitution $t = \tan \frac{x}{2}$ to evaluate $\int_0^{\frac{\pi}{2}} \frac{dx}{2 \sin x + \cos x + 1}$ | 4 |

Question 2 (15 marks)

- | | | |
|---|---|--------|
| a | By first writing $1 + i\sqrt{3}$ in mod-arg form, express $(1 + i\sqrt{3})^7$ in the form $a + ib$ | 2 |
| b | Find the square roots of $40 + 42i$ | 2 |
| c | Let $z = \frac{3 + 4i}{5}$ and $w = \frac{12 + 5i}{13}$ so that $ z = w = 1$
(i) Write zw and $\bar{z}\bar{w}$ in the form $x + iy$
(ii) Hence by considering the square of the moduli find different pairs of positive integers p and q such that $p^2 + q^2 = 65^2$ | 2 |
| d | On separate Argand Diagrams sketch
(i) $ z - 4 = z + 4i $
(ii) $\arg(z - 4) = \arg(z + 4i)$ | 1
1 |
| e | In an Argand Diagram, OABCDE (in clockwise order) is a regular hexagon, where O is the origin and A represents the number $4i$.
(i) Sketch the figure stating the numbers represented by B, C and E.
(ii) The figure is rotated anticlockwise through 90° about A to give a figure A'B'C'D'E'O. Sketch the figure stating the numbers represented by B', C' and E'. | 2
3 |

Question 3 (15 marks)



a The diagram shows the graph of $y = f(x)$. Draw separate one-third page sketches of the graphs of the following:

- (i) $y = \frac{1}{f(x)}$ 2
- (ii) $y^2 = f(x)$ 2
- (iii) $y = \ln f(x)$ 2
- (iv) $|y| = f(|x|)$ 2

b Sketch the following graphs showing features such as asymptotes, intercepts and turning points.

- (i) $y = \frac{x-1}{(x-5)(x+2)}$ 4
- (ii) $y = \ln(\sin e^x)$ 3

Question 4 (15 marks)

a Given that $\alpha, \beta, \gamma, \delta$ are the roots of $x^4 - 2x^3 - 5x^2 + x + 7 = 0$, find

- (i) $\Sigma \alpha$ 2
- (ii) $\Sigma \alpha\beta$ 1,2,1
- (iii) $\Sigma \alpha\beta\gamma$ 2
- (iv) $\alpha\beta\gamma\delta$ 2
- (v) $\Sigma \alpha^2$ 2
- (vi) $\Sigma \alpha^3$ 2
- (vii) $\Sigma \alpha^4$ 2
- (viii) the equation with roots $\frac{2}{\alpha}, \frac{2}{\beta}, \frac{2}{\gamma}, \frac{2}{\delta}$ 2

b If ω is a complex cube root of unity, show that

- (i) ω^2 is the other complex root 1
- (ii) $1 + \omega + \omega^2 = 0$ 1
- (iii) $(a + b + c)(a + b\omega + c\omega^2)(a + b\omega^2 + c\omega) = a^3 + b^3 + c^3 - 3abc$ 2

c Factorise $x^6 - 7x^2 + 6$ over the

- (i) rational numbers 1,1,1
- (ii) real numbers 1,1,1
- (iii) complex numbers 1,1,1

Question 5 (15 marks)

- a (i) On the same axes sketch $y = \sin 2x$ and $y = \tan x$ for $|x| \leq \pi/2$ 2
- (ii) Find the area bounded by these curves 2

(iii) Find the volume generated when these regions are rotated about the x axis. 3

(iv) Use the foregoing results to find the volume if these regions are rotated about the line $x = -1$ 1

b A solid has an elliptical base with equation $16x^2 + 25y^2 = 400$. Each vertical cross section perpendicular to the x axis is in the shape of a parabola (vertex uppermost) with its latus rectum in the base. Clearly explaining your method, find the volume of the solid. 7

Question 6 (15 marks)

a If P is any point on the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$, show that the difference of the distances of P from the two foci is a constant and thus independent of the position of P. Give the value of this constant. 2

b The tangent at P ($a \cos \theta, b \sin \theta$) on an ellipse meets the coordinate axes at Q and R. 3

(i) Find the area of the triangle OQR and deduce its least value. 2

(ii) Hence state the minimum area of a quadrilateral which circumscribes the ellipse and sketch TWO quadrilaterals with that property. 1

(iii) Formulate a statement which relates the ratio of the area of an ellipse and that of a circumscribing parallelogram to the ratio of the area of a circle and that of the circumscribing square. 1

c (i) Find the equation of the normal to the curve $xy = c^2$ at P ($cp, \frac{c}{p}$) 2

(ii) This normal meets the x axis at Q. Show that the coordinates of M, the midpoint of PQ, are $(\frac{c(2p^4 - 1)}{2p^3}, \frac{c}{2p})$ 2

(iii) Find the equation of the locus of M. 3

Question 7 (15 marks)

a Two light rigid rods AB and BC, each of length $2m$, are smoothly jointed at B and the rod AB is smoothly jointed at A to a smooth vertical rod. The joint at B has a mass of 3 kg attached. A ring of mass 2 kg is smoothly jointed to BC at C and can slide on the vertical rod below A. The ring rests on a smooth horizontal table fixed to the rod $2\sqrt{3} \text{ m}$ below A. The system rotates about the rod with an angular velocity ω .

- (i) Find the forces in the rods and the force exerted on the ring by the table. (Draw a clear diagram) 5
- (ii) What value must ω exceed for the ring to rise above the table? 2

b A body of mass m falling under gravity experiences a resistance per unit mass of $k v$ where v is its velocity. Leaving your answer in terms of g :-

- (i) Find the terminal velocity of the body. 1
- (ii) Find the time taken and how far it falls to attain half of this terminal velocity. (Clearly define your notation) 3,4

Question 8 (15 marks)

a Two equal circles touch at A. AB is a diameter of one circle. BR is the tangent from B to the other circle and cuts the first circle at Q. Find the ratio of BQ:QR. 3

b (i) Sketch the quadrilateral $|x-2|+|y|=1$ and calculate its area 2

(ii) Use the method of cylindrical shells to find the volume generated when this area is rotated around the y axis 3

c Let $I_n = \int_0^{\pi/2} \cos^n x dx$

- (i) Show that $I_n = \frac{n-1}{n} I_{n-2}$, for $n \geq 2$. 3
- (ii) Hence show that $\int_0^{\pi/2} \cos^{2n} x dx = \frac{\pi(2n)!}{2^{2n+1}(n!)^2}$ 4