Scots 2004 Maths. Ext. 2 Trial

Question 1 **MARKS**

(a) Find the exact value of:

(i) $\int_0^2 \frac{x^3}{x^2+2}dx$ [2]

(ii) $\int_0^{\frac{\pi}{2}} \frac{\cos x}{\sqrt{1+\sin x}}dx$ [2]

(b) Find $\int \frac{1}{5+4\cos 2\theta}d\theta$. [4]

(c) Given that $I_n = \int_0^{\frac{\pi}{4}} \sec x \tan^n x dx$, $n = 1, 2, 3 \ldots$

(i) Find I_1. [1]

(ii) Prove that $I_n = \frac{\sqrt{2}}{n} - \frac{n-1}{n} I_{n-2}$ [4]

(iii) Evaluate I_5. [2]

Question 2 (Start a new booklet) **MARKS**

(a) Find the equation of the tangent to the curve $x^3 + y^3 - 8y + 7 = 0$ at the point (1, 2) [3]

(b) Draw a neat sketch of the function $f(x) = x^2 - c^2$, where c is a positive constant. State the coordinates of its vertex and its points of intersection with both coordinate axes. [1]

(c) Hence, on separate diagrams, draw neat sketches of the following:

(i) $y = \frac{1}{x^2 - c^2}$ [2]

(ii) $y = \left|\frac{1}{x^2 - c^2}\right|$ [1]

(iii) $y^2 = \frac{1}{x^2 - c^2}$ [2]

(d) Given that $f(x) = \frac{(5-x)(1+x)}{5}$ and $h(x) = \ln\{f(x)\}$:

(i) State the largest possible domain of $y = h(x)$. [1]

(ii) Sketch the graph of $y = h(x)$. [2]

(iii) Find the equation of the inverse function $y = h^{-1}(x)$. [2]

(iv) Find the domain of the inverse function $y = h^{-1}(x)$. [1]

Question 3 (Start a new booklet) **MARKS**

(a) $z^2 + (1+i)z + k = 0$ has $1-2i$ as a root.

(i) Find the other root in the form $a+ib$. [1]

(ii) Find the value of k. [1]

(b) On separate Argand Diagrams, sketch the locus of z described by each of the following condition:

(i) $\arg\left(\frac{z}{z+2}\right) = \frac{\pi}{4}$ [3]

(ii) $|z-2| = 3|z+2i|$ [3]

(iii) $z\bar{z} = z + \bar{z}$ [2]

(c) A sequence of complex numbers z_n is given by the rule $z_1 = w$ and $z_n = c(\overline{z_{n-1}})$ for $n = 1, 2, 3, \ldots$, where w is a given complex number and c is a complex number with modulus 1. Show that $z_3 = w$. [2]

(d) (i) Express $z_1 = \sqrt{2} - i\sqrt{2}$ in modulus-argument form. [1]

(ii) On an Argand Diagram OPQ is an equilateral triangle where O is the origin, P is the point representing the complex number z_1 (as in part (i)) and Q is a point in the first quadrant representing the complex number z_2.
Express z_2 in the form $z_2 = a + ib$, where a and b are real numbers. [2]

Question 4 (Start a new page) **MARKS**

(a) (i) Show that $1+i$ is a root of the equation
$z^4 - 6z^3 + 15z^2 - 18z + 10 = 0$. [2]

(ii) Hence, solve this equation over the field of complex numbers. [3]

(b) The roots of the equation $x^3 + 7x^2 - 5x - 1 = 0$ are α, β and γ.

(i) Evaluate $\alpha^2 + \beta^2 + \gamma^2$. [2]

(ii) Evaluate $\alpha^3 + \beta^3 + \gamma^3$. [2]

(iii) Find the cubic equation with roots $\alpha^2, \beta^2, \gamma^2$. [2]

(c) $P(x) \equiv (x+c)^4 - 32x$ and $P(x)$ has a double root at $x = \alpha$.

(i) Prove that $\alpha = 2 - c$. [2]

(ii) Find the numerical values of α and c. [2]

Question 5 (Start a new booklet) **MARKS**

(a) Sketch the following hyperbola, showing the foci, the directrices and the asymptotes: $\frac{y^2}{9} - \frac{x^2}{4} = 1$. [3]

(b) H is the hyperbola $2xy = a^2$

(i) Prove that, for all values of t, the point $P\left(\frac{at}{2}, \frac{a}{t}\right)$ lies on H and that the tangent at P has equation $2x + t^2 y = 2at$. [3]

(ii) S is the point (a, a) and the perpendicular from S to the above tangent meets the tangent at T. Prove that the line ST has equation $t^2 x - 2y = at^2 - 2a$. [2]

(iii) Prove that, as P moves on the hyperbola, the locus of T is a circle centred at the origin. [2]

(c) P is a point on the ellipse $4x^2 + 5y^2 = 20$. The curve has S and S' as its foci. Show that the sum of the distances from P to the foci is independent of the position of P. [5]

Question 6 (Start a new booklet) **MARKS**

(a) Consider the curve $f(x) = x^2(a - x)$, $0 \le x \le a$.

(i) Sketch the graph of $y = f(x)$. [1]

(ii) S is the region enclosed by $y = f(x)$ and the x axis. A vertical strip of S, width δx, at a distance of x from the y axis, revolves about the y axis to form a cylindrical shell. Show that, if δx is small, the volume, δV, of the shell is approximately $2\pi(ax^3 - x^4)\delta x$. [2]

(iii) Find the volume of the solid formed when S revolves about the y axis. [2]

(b) The base of a solid is the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$. Sections perpendicular to the x axis are squares with one side in the base of the solid. [3]
Show that the volume of the solid is $\frac{16ab^2}{3}$ cubic units.

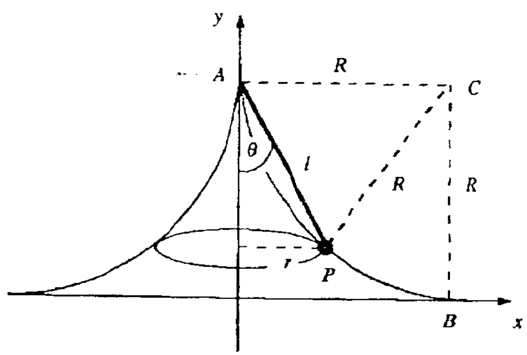
(c) (i) Use De Moivre's Theorem to show that $\cos 3\theta = 4\cos^3\theta - 3\cos\theta$. [2]

(ii) Deduce $8x^3 - 6x - 1 = 0$ has solutions $x = \cos\theta$ where $\cos 3\theta = \frac{1}{2}$. [2]

(iii) Find the roots of $8x^3 - 6x - 1 = 0$ in the form of $\cos\theta$ for $0 \le \theta \le \pi$. [2]

(iv) Hence evaluate $\cos\frac{\pi}{9}\cos\frac{2\pi}{9}\cos\frac{4\pi}{9}$. [1]

Question 7 (Start a new booklet) **MARKS**



AB is an arc of a circle centre C and radius R. A smooth surface is formed by rotating the arc AB through one revolution about the y axis. A light, inextensible string of length l, $l \le R$, is attached to point A, and a particle of mass m is attached to the other end. The particle is set in motion, tracing out a horizontal circle on the surface with constant angular velocity ω radians per second, while the string stays taut.

(i) When the particle is in the position P shown in the diagram, explain why the direction of the force N exerted by the surface on the particle is towards C. [1]

(ii) If the string makes an angle θ with the vertical, show that $\angle ACP = 2\theta$. [1]

(iii) Show on a diagram the tension force T, the force N and the weight force of magnitude mg acting on the particle, indicating their directions in terms of θ. [2]

(iv) By resolving forces show that $N = ml\sin\theta\left(\frac{g}{l}\sec\theta - \omega^2\right)$. [4]

(v) Deduce that there is a maximum value ω for the motion to occur as described and write down this maximum value. [2]

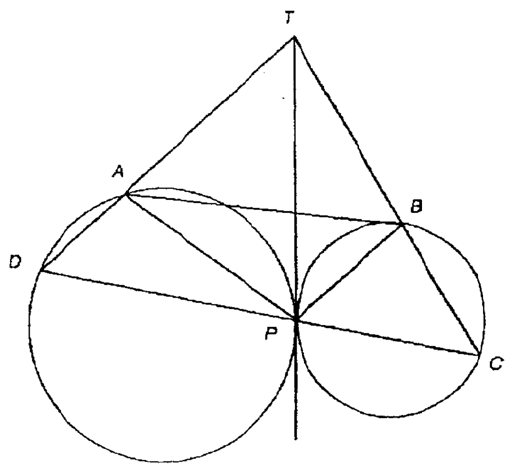
Question 7 Continued **MARKS**

(vi) If $l = R$, find T in terms of l, m and ω^2. [3]

(vii) Describe what happens to T and N as ω increases. [2]

Question 8 (Start a new booklet)

(a)



(i) Copy the diagram into your writing booklet. [1]

(ii) Show that ATBP is a cyclic quadrilateral. [4]

Question 8 Continued **MARKS**

(b) The real number x is such that $x^2 = x + 1$. [5]
The Fibonacci sequence of numbers T_n, where $n = 1, 2, 3 \ldots$
is given by $T_1 = 1$, $T_2 = 1$ and $T_n = T_{n-1} + T_{n-2}$, where $n = 3, 4, 5 \ldots$

Use Induction to show that $x^n = T_{n-1} + T_n x$ for all positive integers $n \geq 2$.

(c) By considering $\int_0^3 (1+x)^n \, dx$, prove that [5]

$$\sum_{k=0}^{n} \frac{1}{k+1} {}^nC_k 3^{k+1} = \frac{1}{n+1}(4^{n+1} - 1)$$