

QUESTION 1 (15 marks)

Scot 2008 Ext 2 trial

a) Evaluate $|3 + 2i|$

1

b) i) If $v = \frac{1+i\sqrt{3}}{2}$ show that $v^3 = -1$.

2

ii) Hence calculate v^{10} .

2

c) If z is a complex number so that $|z| = 2$ and $\arg z = \frac{\pi}{6}$, mark clearly on the same Argand diagram the points representing the complex numbers:

i) z ii) iz iii) \bar{z} iv) $\frac{1}{z}$ v) $z\bar{z}$ vi) z^2 vii) $z^2 + z$ viii) $z^2 - z$ 10

QUESTION 2 (15 marks)

a) Find $\int \frac{dx}{x^2 - 6x + 13}$

2

b) Find $\int \tan x \sec^2 x \, dx$

2

c) i) Show that $f(x) = \sin^{-1}x$ is an odd function.

2

ii) Hence or otherwise find $\int_{-1}^1 (\sin^{-1} x)^3 \, dx$

1

d) $\int_0^{\sqrt{2}} \sqrt{4-x^2} \, dx$

4

e) $\int e^x \cos x \, dx$

4

QUESTION 3 (10 marks)

a) Use the method of cylindrical shells to find the volume of the solid (paraboloid) obtained when the region between the curve $y = \frac{1}{2}\sqrt{x-2}$, the x-axis and the line $x = 6$ is rotated about the x axis. 4

b) Prove $\frac{d^2x}{dt^2} = \frac{d}{dx}\left(\frac{1}{2}v^2\right)$, where x denotes displacement, and v denotes velocity. 2

c) The acceleration of a particle moving in a straight line is given by $\ddot{x} = xe^x$, where x is the displacement from 0. The particle is initially at rest.

The particle starts at $x = 0$.

i) Prove that $v^2 = 2e^x(x-1) + 2$ 3

ii) Describe the subsequent motion of the particle after it leaves the origin and explain why the particle can only move in one direction 1

QUESTION 4 (18 marks)

a) The equation $x^3 - x^2 - 3x + 2 = 0$ has roots α, β, γ . Find the monic polynomial equation with roots $\alpha^2, \beta^2, \gamma^2$. 4

b) If $x = \alpha$ is a double root of the equation $P(x) = 0$, show that $x = \alpha$ is a root of the equation $P'(x) = 0$. 4

c) i) Show that $1+i$ is a root of the polynomial $Q(x) = x^3 + x^2 - 4x + 6$ 2

ii) hence resolve $Q(x)$ into irreducible factors over the complex number field. 3

d) If α, β, γ are the roots of the cubic equation $x^3 + qx + r = 0$, prove that $(\beta - \gamma)^2 + (\gamma - \alpha)^2 + (\alpha - \beta)^2 = -6q$. 5

QUESTION 5 (18 marks)

The ellipse \mathcal{E} has the cartesian equation $\frac{x^2}{25} + \frac{y^2}{16} = 1$.

- i) Write down the eccentricity 1
- ii) Write down the coordinates of the foci S and S' 1
- iii) Write down the equations of the directrices. 1
- iv) Sketch the ellipse \mathcal{E} . 1
- v) Show that any point P on \mathcal{E} can be represented by the coordinates $(5 \cos \theta, 4 \sin \theta)$ 1
- vi) Prove that $PS + PS'$ is independent of the position of P on the ellipse \mathcal{E} . 3
- vii) Show that the equation of the normal N at the point P on the ellipse \mathcal{E} is 2
 $5 \sin \theta x - 4 \cos \theta y = 9 \sin \theta \cos \theta$
- viii) If this normal meets the major axis of the ellipse in M and the minor axis in N ,
prove that $\frac{PM}{PN} = \frac{16}{25}$. 3
- ix) Also show that the line PN bisects the angle $S'PS$. 5

QUESTION 6 (14 marks)

- i) By considering the curve $g(x) = x^6 - 4x^5 + 4x^4$, sketch the graph of $f(x) = x^6 - 4x^5 + 4x^4 - 1$ showing that it has 4 real zeroes. 4

On different diagrams sketch the curves:

- ii) $y = |f(x)|$ 2
- iii) $y = f(|x|)$ 2
- iv) $y^2 = f(x)$ 3
- v) Calculate the slope of the curve $y^2 = f(x)$ at any point x and describe the nature of the curve at a zero of $f(x)$. 3

QUESTION 7 (15 marks)

a) A parachutist of M kilograms is dropped from a stationary helicopter of height H metres above the ground. The parachutist experiences air resistance during its fall equal to MkV^2 , where V is its velocity in metres per second and k is a positive constant. Let x be the distance in metres of the parachutist from the helicopter, measured positively as it falls.

i) Show that the equation of motion of the parachutist is $\ddot{x} = g - kV^2$, where g is the acceleration due to gravity. 1

ii) Find V^2 as a function of x . 4

iii) Find the velocity U of the parachutist as he hits the ground in terms of g , k and H . 1

iv) Find the velocity of the parachutist as he hits the ground if air resistance is neglected. 2

b)

i) Prove the identity $\cos 3A = 4\cos^3 A - 3\cos A$ 2

ii) Show that $x = 2\sqrt{2} \cos A$ is a root of the equation $x^3 - 6x + 2 = 0$ provided that 2

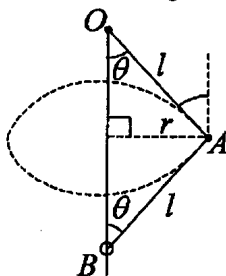
$$\cos 3A = -\frac{1}{2\sqrt{2}}$$

iii) Find the three roots of the equation $x^3 - 6x + 2 = 0$, using the results from part (ii) above. Give your answer to three decimal places. 3

QUESTION 8 (15 marks)

a) A particle A of mass $2m$ is attached by a light inextensible string of length l to a fixed point O and is also attached by another light inextensible string of the same length to a small ring B of mass $3m$ which can slide on a fixed smooth vertical wire passing through O . The particle A describes a horizontal circle of radius r , and OA is inclined at an angle $\theta = \frac{\pi}{3}$ with the downward vertical.

Dimension diagram



$$\theta = \frac{\pi}{3}$$

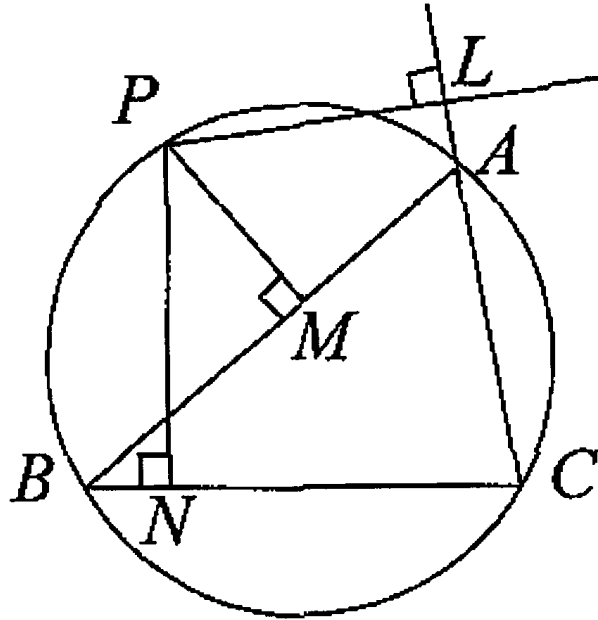
i) Find the tension in the strings OA and AB 5

ii) Find the angular velocity of A . 3

iii) Describe what happens to the system as the angular velocity increases. 1

b) $\triangle ABC$ is a triangle inscribed in the circle. P is a point on the minor arc AB . The points L , M , and N are the feet of the perpendiculars from P to CA produced, AB , and BC respectively.

Copy the diagram into your answer booklet and show that L , M and N are collinear.



END OF EXAM