

THE SCOTS COLLEGE

2009

**TRIAL HIGHER SCHOOL CERTIFICATE
EXAMINATION**

MATHEMATICS EXTENSION 2

General Instructions

- All questions are of equal value
- 5 minutes reading time
- Working time - 3 hours
- Write using blue or black pen
- Board approved calculators may be used
- Start a new booklet for each question
- All necessary working should be shown in every question

Standard Integrals Table is attached

TOTAL MARKS: 120

WEIGHTING: 40 %

a. If $z_1 = 1 + 3i$, $z_2 = 1 - i$,

(i) Find in the form $a + ib$, where a and b are real, the numbers $z_1 \times z_2$ [2]
and $\frac{z_1}{z_2}$.

(ii) On an Argand Diagram the vectors \overline{OA} , \overline{OB} represent the complex [1]
numbers $z_1 z_2$ and $\frac{z_1}{z_2}$ respectively (where z_1 and z_2 are given above).

Show this on an Argand Diagram, giving the coordinates of A and B .

(iii) From your diagram, deduce that $\frac{z_1}{z_2} - z_1 z_2$ is real. [1]

b. Given that $z = \sqrt{3} - i$,

(i) Express z in modulus-argument form. [1]

(ii) Hence, evaluate the following in the form $x + iy$:

(α) z^5 [1]

(β) $(\overline{z})^5$ [1]

(γ) $\frac{z^5}{(\overline{z})^5}$ [1]

c. On an Argand diagram, sketch the locus of z if:

(i) $|z + 3| < |z - 1 - 4i|$ [2]

(ii) $4 \arg \frac{z-1}{z+3} = \pi$ [2]

d. Solve the equation $\frac{1}{z+1} + \frac{1}{1-2i} = 1$, [3]

- a. Let $f(x) = \frac{4}{x} - x$. Provide separate half page sketches of the graphs of the following:

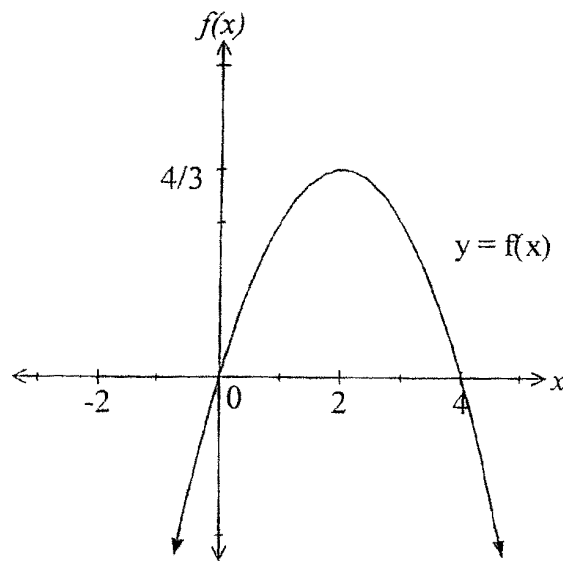
(i) $y = f(x)$ [2]

(ii) $y = \sqrt{f(x)}$ [2]

(iii) $y = e^{f(x)}$ [2]

Label each graph carefully.

b.



(i) Use the diagram to find the values of a, b, c given $f(x) = ax^2 + bx + c$ [2]

(ii) Solve $-1 \leq f(x) \leq 1$ [3]

(iii) Hence or otherwise sketch

α $y = \ln[f(x)]$ [2]

β $y = \cos^{-1}[f(x)]$ [2]

- a. A particle moves in a straight line. Prove that its acceleration at any instant can be expressed as: [2]

(i) $v \frac{dv}{dx}$

(ii) $\frac{d}{dx} \left(\frac{1}{2} v^2 \right)$

where x denotes its position and v its velocity at any instant.

- b. A particle moving in a straight line from the origin is subject to a resisting force which produces a retardation of kv^3 where v is the speed at time t and k is a constant. If u is the initial speed, x is the distance moved in time t ,

(i) Show that $v = \frac{u}{kux + 1}$ [3]

(ii) Deduce that $kx^2 = 2t - \frac{2x}{u}$. [2]

- c. A particle of mass m kg is projected upwards with initial speed U m/s. The particle is subject to gravity and a resistive force of magnitude mkv Newtons, where v m/s is the speed of the particle at any instant and k is a constant, $k > 0$.

(i) Explain why $\ddot{x} = -g - kv$, where g is the acceleration due to gravity. [2]

(ii) Prove that the particle will reach a maximum height after T seconds [3]

$$\text{given by } T = \frac{1}{k} \ln \left(\frac{g + kU}{g} \right).$$

(iii) Prove that the maximum height reached is $\frac{1}{k}(U - gT)$ metres. [3]

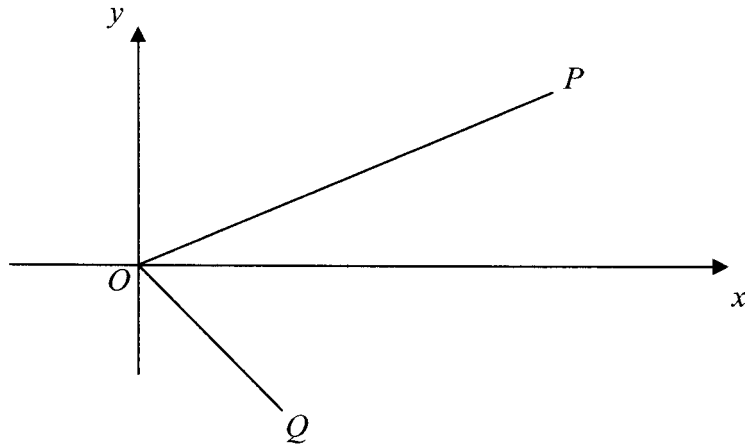
QUESTION 4**[15 MARKS]****START A NEW BOOKLET****MARKS**

- a.** The area bounded by $y = e^{2x}$, the x -axis, the y -axis, and the line $x = 2$, is rotated about the y -axis. Use the method of cylindrical shells to find the volume of the solid formed. **[5]**
- b.** The area bounded by the parabola $y = 2x^2$ and the line $y = 8$ is rotated about the line $y = 10$. Use the slice (washer) method to find the volume of the solid formed. **[4]**
- c.** The horizontal base of a solid is the circle $x^2 + y^2 = 16$. Vertical cross-sections perpendicular to the x -axis are regions bounded by a parabola and its latus rectum, with the latus rectum lying on the base and the vertex of the parabola vertically above the latus rectum. **[6]**
- (i)** Prove that the area bounded by a parabola of focal length a units and its latus rectum is $\frac{8a^2}{3}$ unit².
- (ii)** Hence find the volume of the solid.

QUESTION 5**[15 MARKS]****START A NEW BOOKLET**

- a.** Prove that the 5 roots of $z^5 = -1$ are:
- (i)** $z = \text{cis} \frac{(2k+1)\pi}{5}$ for $k = 0, 1, 2, 3, 4$. **[3]**
- (ii)** Hence show that $\cos \frac{\pi}{5} + \cos \frac{3\pi}{5} = \frac{1}{2}$ **[2]**
- b.** If the polynomial $P(x) = x^3 + qx + r$ has roots α, β and γ form an equation with roots $\frac{1}{\alpha}, \frac{1}{\beta}$ and $\frac{1}{\gamma}$. **[3]**
- c.** If α, β and γ are the roots of $x^3 - 4x^2 + 5x + 3 = 0$ evaluate $\alpha^3 + \beta^3 + \gamma^3$. **[3]**
- d.** If the three roots of the equation $8x^3 - 36x^2 + 38x - 3 = 0$ are in arithmetic progress, find these roots in ascending order of magnitude. **[4]**

a.



In the Argand Diagram above, point P corresponds to complex number z . The triangle OPQ is a right-angled triangle and $OP = 3OQ$.

- (i) What is the complex number that corresponds to point Q ? [1]
- (ii) $QOPR$ is a rectangle. Write down the complex number that corresponds to R . [2]
- b. (i) Prove the identity $\cos(a-b)x - \cos(a+b)x = 2\sin ax \sin bx$ [2]
- (ii) Hence find $\int \sin 3x \sin 2x dx$ [2]
- c. If $u_1 = 8$, $u_2 = 20$ and $u_n = 4u_{n-1} - 4u_{n-2}$ for $n \geq 3$.
- (i) Determine u_3 and u_4 . [1]
- (ii) Prove by induction that $u_n = (n+3)2^n$ for $n \geq 1$. [4]
- d. If $ax^3 + bx^2 + d = 0$ ($a, b, d \neq 0$) has a double root, show that $27a^2d + 4b^3 = 0$. [3]

a. $\int \frac{dx}{\sqrt{4-9x^2}}$ [2]

b. Find: [4]

(i) $\int xe^x dx$

(ii) $\int \cos^5 x \sin^3 x dx$

c. Find the value of: $\int_0^{\sqrt{2}} \frac{x^3}{x^2+4} dx$ [2]

d. Decompose $\frac{2(x+1)}{(x-1)(2x-1)}$ into partial fractions and hence find $\int_2^5 \frac{2(x+1)}{(x-1)(2x-1)} dx$ [3]

e. [4]

(i) Prove that if $u_n = \int_0^{\frac{\pi}{2}} x^n \sin x dx$, $n \geq 0$, then $u_n + n(n-1)u_{n-2} = n\left(\frac{\pi}{2}\right)^{n-1}$, $n \geq 2$, and

(ii) deduce the value of $\int_0^{\frac{\pi}{2}} x^4 \sin x dx$.

QUESTION 8 [15 MARKS] START A NEW BOOKLET MARKS

a. Use mathematical induction to show that:

$$\sqrt{1} + \sqrt{2} + \sqrt{3} + \dots + \sqrt{n} \leq \frac{4n+3}{6} \sqrt{n} \text{ for all integers } n \geq 1. \quad \text{[5]}$$

b. The normal at $P\left(ct, \frac{c}{t}\right)$ to the hyperbola $xy = c^2$ meets the curve again at Q .

(i) Prove that the equation of the normal is $t^3x - ty = ct^4 - c$ [2]

(ii) Find the coordinates of Q . [2]

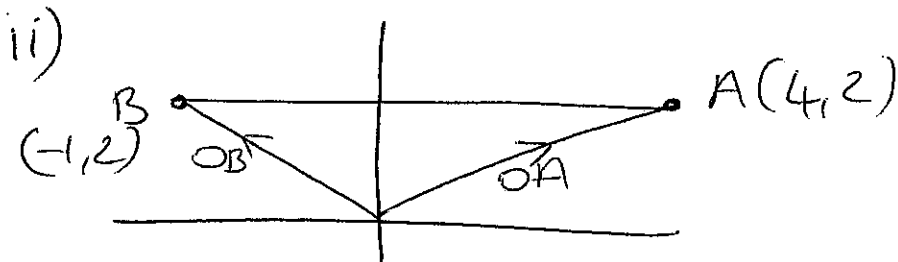
(iii) A line from P through the origin meets the hyperbola again at R . Prove that PR is perpendicular to QR . [3]

(iv) If M is the midpoint of PQ , find the equation of the locus M . [3]

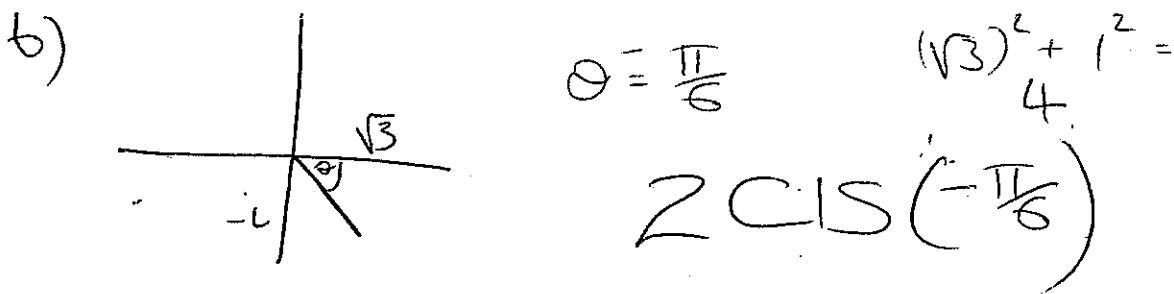
Question 4

a) i) $Z_1 Z_2 = (1+3i)(1-i)$
 $= 4+2i$

ii) $\frac{Z_1}{Z_2} = \frac{1+3i}{1-i} \times \frac{1+i}{1+i} = -1+2i$



$\frac{Z_1}{Z_2} = Z_1 \times Z_2^{-1}$ is the same as BA
 $= -5$



ii)

$$Z^5 = 2^5 \text{ CIS } \left(-\frac{5\pi}{6} \right)$$

$$= 32 \text{ CIS } \left(-\frac{5\pi}{6} \right)$$

$$= 32 \left[-\frac{\sqrt{3}}{2} - \frac{1}{2}i \right]$$

$$= -16(\sqrt{3} + i)$$

$$\bar{Z} = 2 \text{ CIS } \left(\frac{\pi}{6} \right)$$

$$(\bar{Z})^5 = 32 \left[-\frac{\sqrt{3}}{2} + \frac{1}{2}i \right]$$

$$= 16(-\sqrt{3} + i)$$

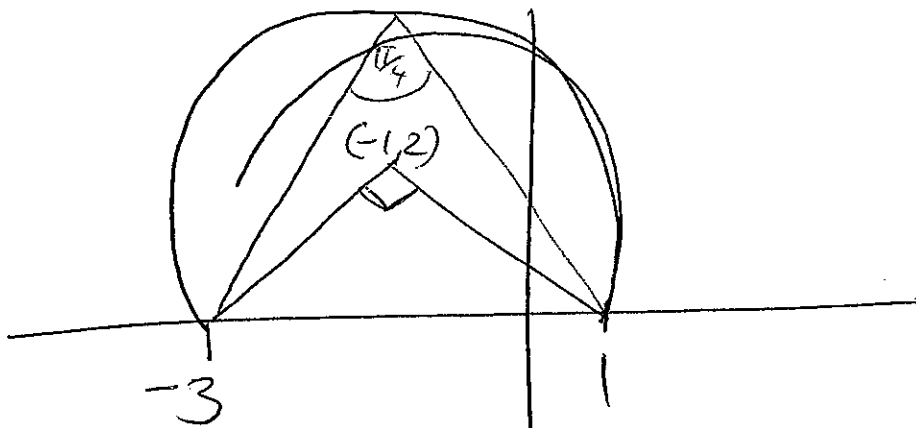
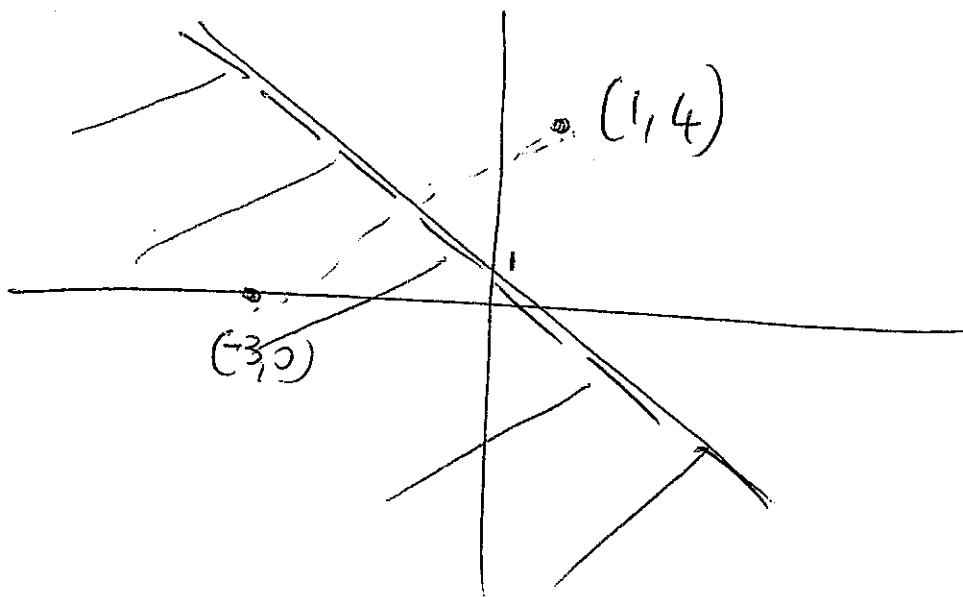
$$\frac{Z^5}{(\bar{Z})^5} = \frac{-16(\sqrt{3} + i)}{16(-\sqrt{3} + i)}$$

$$\frac{\sqrt{3} + i}{-\sqrt{3} + i} \times \frac{-\sqrt{3} - i}{-\sqrt{3} - i}$$

$$\frac{1}{2} + i\frac{\sqrt{3}}{2}$$

Question 5

c)



d)

~~$$\frac{1}{z+1} + \frac{1}{1-2i} = 1$$~~

~~$$\frac{1}{(x+1)+yi} + \frac{1}{1-2i} = 1$$~~

~~$$\frac{1}{(x+1)+yi} \cdot \frac{(x+1)-yi}{(x+1)-yi} + \frac{1}{1-2i} \cdot \frac{(1+2i)}{(1+2i)} = 1$$~~

~~$$\frac{(x+1)-yi}{(x+1)^2+y^2} + \frac{(1+2i)}{5} = 1$$~~

$$\frac{1}{z+1} + \frac{1}{(1-2i)} = 1$$

$$\begin{aligned}\frac{1}{z+1} &= 1 - \frac{(1+2i)}{5} \\ &= 1 - \frac{1}{5} + \frac{2}{5}i\end{aligned}$$

$$\frac{1}{z+1} = \frac{-4 + 2i}{5}$$

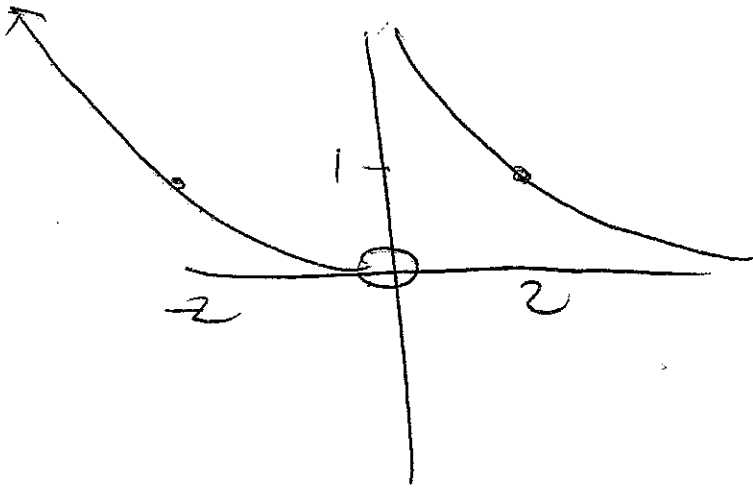
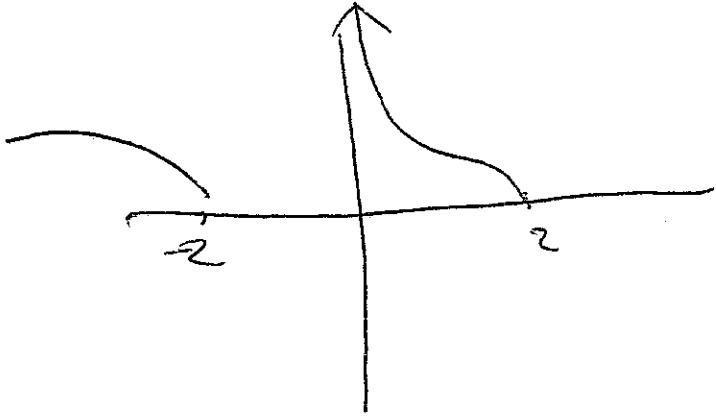
$$z+1 = \frac{-5}{4} \times \frac{5}{-4+2i} \times \frac{(-4-2i)}{(-4-2i)}$$

$$z+1 = \frac{-20-10i}{20}$$

$$z+1 = -1 - \frac{1}{2}i$$

$$z = -2 - \frac{1}{2}i$$

Question 2



Question 8

i) $y = ax^2 + bx + c$

$c = 0$ By observation

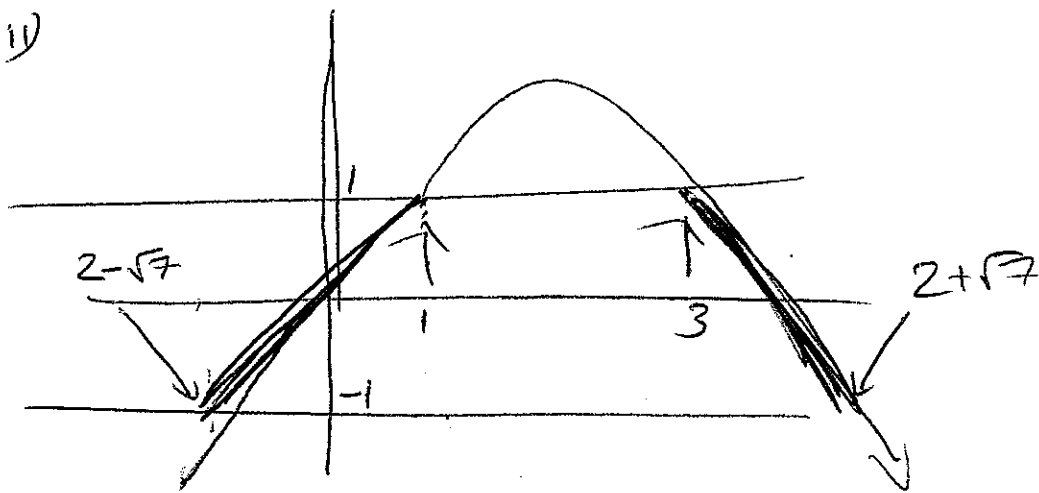
$x = 4 \quad y = 0 \quad 16a + 4b = 0$

$x = 2 \quad y = \frac{4}{3} \quad 2a + b = \frac{2}{3}$

$a = -\frac{1}{3} \quad b = \frac{4}{3}$

$y = -\frac{x}{3}(x-4)$

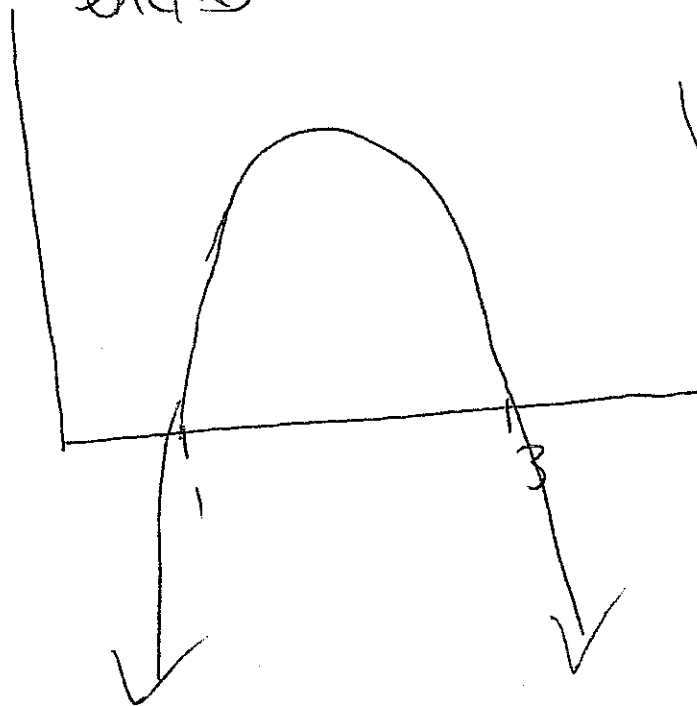
ii)



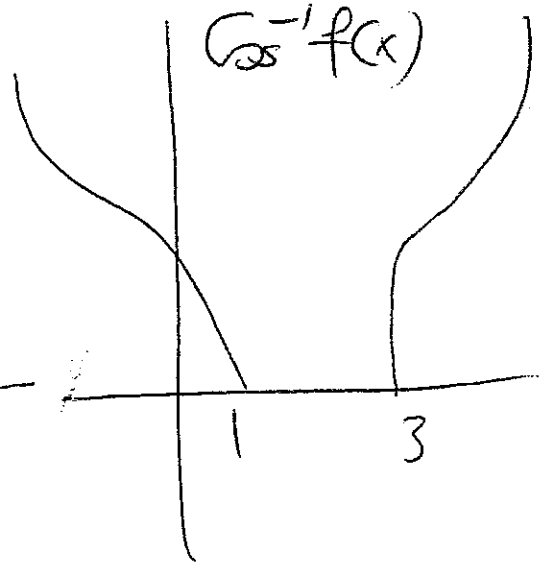
$2 - \sqrt{7} \leq x \leq 1 \quad \text{OR} \quad 3 \leq x \leq 2 + \sqrt{7}$

iii)

$\ln(f(x))$



$\cos^{-1}f(x)$



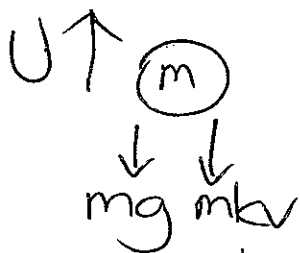
Question B

$$\begin{aligned}
 \text{a) } \frac{dv}{dt} &= \frac{dv}{dx} \cdot \frac{dx}{dt} \\
 &= \frac{dx}{dt} \frac{dv}{dx} \\
 \ddot{x} &= v \frac{dv}{dx} \quad (1)
 \end{aligned}
 \left. \vphantom{\frac{dv}{dt}} \right\}
 \begin{aligned}
 \frac{d}{dx} \frac{1}{2} v^2 &= \frac{1}{2} \cdot 2v \cdot \frac{dv}{dx} \\
 &= v \frac{dv}{dx} \\
 &= \ddot{x} \quad (1)
 \end{aligned}$$

b)

i) Force = $m\ddot{x}$

$$\begin{aligned}
 \therefore m\ddot{x} &= -mg - kv \\
 &= -g - kv \quad (2)
 \end{aligned}$$



ii) $\frac{dv}{dt} = -g - kv$

$$\frac{dt}{dv} = \frac{-1}{g + kv}$$

$$t = -\frac{1}{k} \ln(g + kv) + C \quad (4)$$

when $t=0$ $v=U$

$$\therefore 0 = -\frac{1}{k} \ln(g + kU) + C$$

$$\therefore C = \frac{1}{k} \ln(g + kU)$$

$$\therefore t = \frac{1}{k} [\ln(g + kU) - \ln(g + kv)]$$

$$t = \frac{1}{k} \ln \left[\frac{g + kU}{g + kv} \right]$$

At max height $v=0$ $t_{\text{max height}} = \frac{1}{k} \ln \left(\frac{g + kU}{g} \right)$

⊕) iii)

$$V \frac{dv}{dx} = -g - kV$$

$$\frac{dv}{dx} = \frac{-g - kV}{V}$$

$$\therefore \frac{dx}{dv} = \frac{V}{-g - kV}$$

$$\frac{-1/k}{-g - kV} \frac{V}{V + \frac{g}{k}} \quad (4)$$

$$\therefore \frac{dx}{dv} = -\frac{1}{k} + \frac{g/k}{g + kV}$$

$$= \frac{1}{k} \left[\frac{g}{g + kV} - 1 \right]$$

$$X = \frac{1}{k} \left[\frac{g}{k} \ln(g + kV) - V \right] + C$$

$$x=0 \quad v=U$$

$$C = -\frac{1}{k} \left[\frac{g}{k} \ln(g + kU) - U \right]$$

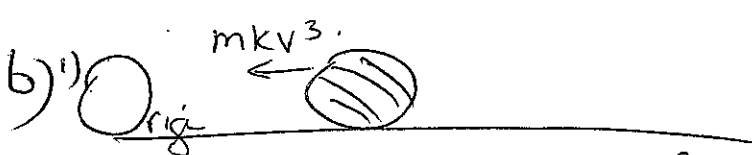
$$X = \frac{1}{k} \left[\frac{g}{k} \ln(g + kV) - \frac{g}{k} \ln(g + kU) - V + U \right]$$

max height $v=0$

↓ Careful

$$\therefore X_{\max} = \frac{1}{k} \left[U - \frac{g}{k} \ln \left[\frac{g + kU}{g + kU} \right] \right]$$

$$X_{\max} = \frac{1}{k} [U - gT]$$



$t=0 \quad x=0$
 $v=U$

$$\ddot{x} = -kV^3$$

$$V \frac{dV}{dx} = -kV^3$$

$$\dots \frac{dV}{dx} = -kV^2$$

$$\dots \frac{dx}{dV} = \frac{-1}{k} V^{-2}$$

$$\dots x = +\frac{1}{k} V^{-1} + C$$

(2)

$$\dots C = -\frac{1}{k} U^{-1}$$

$$x = \frac{1}{k} \left[\frac{1}{V} - \frac{1}{U} \right]$$

$$xk + \frac{1}{U} = \frac{1}{V}$$

$$\frac{kux + 1}{U} = V$$

$$\dots V = \frac{U}{kux + 1}$$

"

$$\frac{dx}{dt} = \frac{U}{kux + 1}$$

$$\frac{dt}{dx} = \frac{kux + 1}{U}$$

(2)

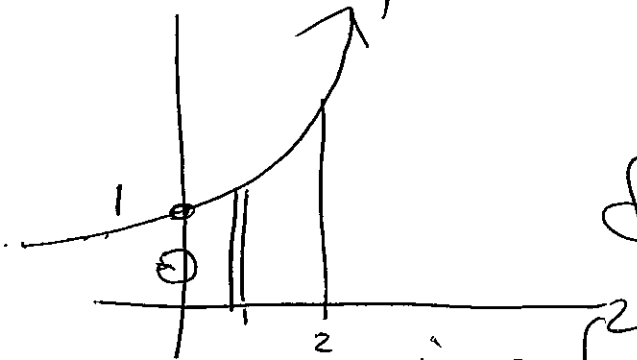
$$t = \frac{kx^2}{2} + \frac{x}{U} + C$$

$$t=0 \quad x=0 \dots C=0$$

$$\dot{2t} = kx^2 + \frac{2x}{U}$$

$$kx^2 = 2t - \frac{2x}{U}$$

Question 2



Height = y
 Radius = x .

$$dV = 2\pi x y dx$$

$$2\pi \int_0^2 x y dx$$

$$I = 2\pi \int_0^2 x e^{2x} dx$$

$$u = x \quad dv = e^{2x}$$

$$du = 1 \quad v = \frac{1}{2} e^{2x}$$

$$I = 2\pi \left[\left[\frac{x e^{2x}}{2} \right]_0^2 - \int_0^2 \frac{e^{2x}}{2} dx \right]$$

$$= 2\pi \left[\frac{2e^4}{2} - \left[\frac{e^{2x}}{4} \right]_0^2 \right]$$

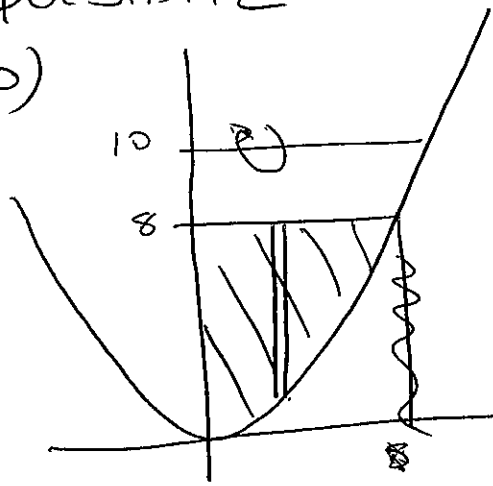
$$= 2\pi \left[2e^4 - \frac{e^4}{4} + \frac{1}{4} \right]$$

$$\text{Volume} = \frac{2\pi}{4} [3e^4 + 1]$$

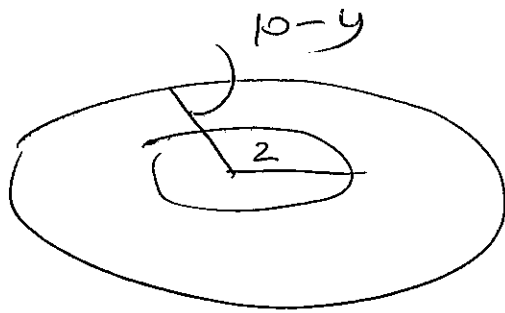
$$\text{Volume} = \frac{\pi}{2} [3e^4 + 1]$$

Question 2

b)



$$\begin{aligned} 8 &= 2x^2 \\ 4 &= x^2 \\ x &= 2 \end{aligned}$$



$$dV = \pi [(10-y)^2 - 2^2] dx$$

$$V = \int_0^2 \pi [100 - 20y + y^2 - 4] dx$$

$$V = \pi \int_0^2 96 - 20y + y^2 dx$$

$$= \pi \int_0^2 96 - 40x^2 + 4x^4 dx$$

$$= \pi \left[96x - \frac{40}{3}x^3 + \frac{4}{5}x^5 \right]_0^2$$

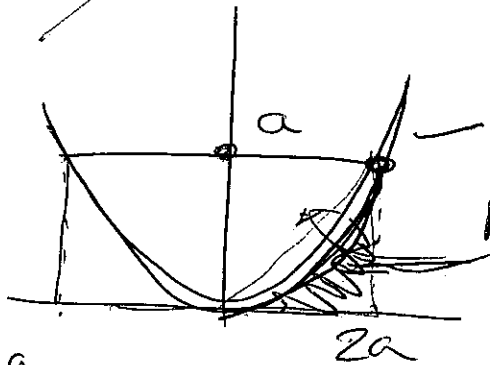
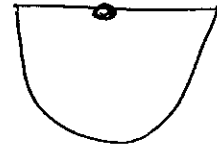
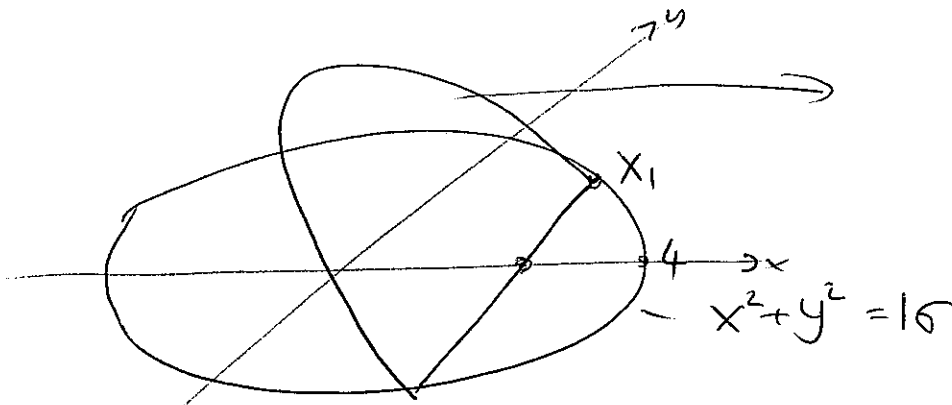
$$= \pi \left[96 \times 2 - \frac{40}{3} \times 8 + \frac{4}{5} \times 32 \right]$$

$$= \frac{\pi}{15} [2880 - 1600 + 384]$$

$$\text{Volume} = \frac{\pi}{15} (16,64) U^3$$

c)

LATUS RECTUM.



(6)

Area = $\int_0^{2a} y dx$

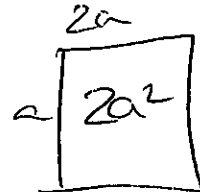
= $\int_0^{2a} \frac{x^2}{4a} dx$

= $\left[\frac{x^3}{12a} \right]_0^{2a} = \frac{2a^2}{3}$

$\int_0^a x dy$
 $\int_0^a 2\sqrt{ay} dy$

$2\sqrt{a} \times \left[\frac{2}{3} y^{3/2} \right]_0^a$

$\frac{4}{3} \sqrt{a} \times a^{3/2}$
 $\frac{4}{3} a^2$



$\therefore \frac{4a^2}{3} - \frac{2a^2}{3} = \frac{2a^2}{3}$

Parabola = $\frac{2a^2}{3}$

$\therefore \text{Area} = \frac{8}{3} a^2 u^3$

ii) $dA = \frac{8}{3} a^2 u^2 du$

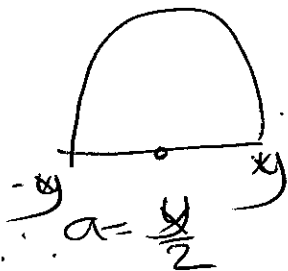
$dV = \frac{8}{3} \frac{u^2}{4} \times dx$

$V = \frac{4}{3} \left[16x - \frac{x^3}{3} \right]_0^4$

$V = \frac{4}{3} \left[64 - \frac{64}{3} \right]$

= $\frac{4}{9} \times 128$

= $\frac{512}{9} u^3$

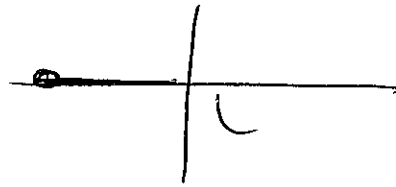


$V = \frac{2 \times 8}{12} \int_0^4 y^2 dx$

= $\frac{4}{3} \int_0^4 (16 - x^2) dx$

Question 5

a) $z^5 = 1$



$$z^5 = 1 \text{cis } \pi$$

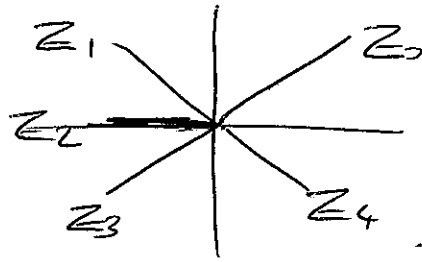
$$z_1 = 1 \text{cis } \frac{\pi}{5},$$

$$z_2 = 1 \text{cis } \left(\frac{\pi}{5} + \frac{2\pi}{5} \right)$$

$$z_3 = 1 \text{cis } \left(\frac{\pi}{5} + \frac{2\pi}{5} + \frac{2\pi}{5} \right)$$

$$z = \text{cis } \frac{(2k+1)\pi}{5} \quad \text{for } k=0,1,2,3,4$$

ii) Roots of $z^5 + 1 = 0$ are z_0, z_1, z_2, z_3, z_4



$$z_3 = -1 \quad z_0 + z_1 + z_2 + z_3 + z_4 = 0$$

$$z_0 + z_1 + z_3 + z_4 = 1$$

$$z_0 + z_3 = \bar{z}_0 \quad z_4 = \bar{z}_1$$

$$z_0 + z_1 + z_3 + z_4 = 1$$

$$\bar{z}_0 + z_0 + \bar{z}_1 + z_1 = 1$$

$$2 \cos \frac{\pi}{5} + 2 \cos \frac{3\pi}{5} = 1$$

$$\cos \frac{\pi}{5} + \cos \frac{3\pi}{5} = \frac{1}{2}$$

b) Question 3

$$P(x) = x^3 + qx + r$$

Roots α, β, γ

$$\therefore P\left(\frac{1}{x}\right) = \left(\frac{1}{x}\right)^3 + q\left(\frac{1}{x}\right) + r$$

$$\frac{1}{x^3} + \frac{q}{x} = -r$$

$$rx^3 + qx^2 + 1 = 0$$

c) $x^3 - 4x^2 + 5x + 3 = 0$

Roots are α, β, γ .

d) $\therefore (x^{1/3})^3 - 4(x^{1/3})^2 + 5(x^{1/3}) + 3 = 0$

$$x - 4(x^{2/3}) + 5(x^{1/3}) + 3 = 0$$

$$x + 3 = 4(x^{2/3}) - 5(x^{1/3})$$

$$(x+3) = x^{1/3}(4x^{1/3} - 5)$$

$$\begin{array}{r} 1 \\ 13 \end{array} \begin{array}{r} 1 \\ 2 \\ 3 \end{array} \begin{array}{r} 1 \\ 3 \end{array}$$

$$x^3 + 9x^2 + 27x + 27 = x[64x +$$

$$(\alpha + \beta + \gamma)^3$$

$$(\alpha + \beta + \gamma)(\alpha + \beta + \gamma)$$

$$(\alpha^2 + 2\alpha\beta + 2\alpha\gamma + \beta^2 + 2\beta\gamma + \gamma^2)(\alpha + \beta + \gamma)$$

$$\alpha^3 + 2\alpha^2\beta + 2\alpha^2\gamma + \alpha\beta^2 + 2\alpha\beta\gamma + \alpha\gamma^2 + 2\alpha\beta^2 + 2\alpha\beta\gamma + \beta^3 + 2\beta^2\gamma + \alpha\beta$$

$$(A + B + C)(A + B + C)$$

$$A^2 + AB + AC + BA + B^2 + BC + CA + CB + C^2$$

$$[A^2 + B^2 + C^2 + 2AB + 2BC + 2AC]$$

$$\cancel{A^3} + AB^2 + AC^2 + \cancel{2A^2B} + 2ABC + \cancel{2A^2C}$$

$$\cancel{A^2B} + \cancel{B^3} + C^2B + 2AB^2 + 2B^2C + 2ABC$$

$$\cancel{A^2C} + B^2C + \cancel{C^3} + 2ABC + 2BC^2 + 2AC^2$$

$$A^3 + B^3 + C^3 + 3A^2B + 3A^2C + 3$$

$$\frac{\partial P}{\partial P} =$$

$$\frac{\partial P}{\partial P} \frac{\partial P}{\partial P} = \frac{\partial P}{\partial P} \quad (10)$$

Question 3
 α, β, γ

or
 $\alpha-d, \alpha, \alpha+d$

$$\therefore \text{Sum of roots } 3\alpha = \frac{36}{8}$$

$$\therefore \alpha = \frac{3}{2}$$

$$\text{Pais } (\alpha-d)\alpha + (\alpha+d)\alpha + (\alpha-d)(\alpha+d)$$
$$\alpha^2 - \alpha d + \alpha^2 + \alpha d + \alpha^2 - d^2$$

$$3\alpha^2 - d^2 = \frac{38}{8}$$

$$\frac{27}{4} - d^2 = \frac{38}{8}$$

$$d^2 = \frac{27}{4} - \frac{38}{8}$$

$$d^2 = \frac{16}{8}$$

$$d^2 = 2$$

$$\therefore d = \sqrt{2}$$

$$\frac{3}{2} - \sqrt{2}, \frac{3}{2}, \frac{3}{2} + \sqrt{2}$$

Check

$$\left(\frac{3}{2} - \sqrt{2}\right)\left(\frac{3}{2} + \sqrt{2}\right)$$

$$\frac{9}{4} - 2$$

$$\frac{1}{4} \times \frac{3}{2}$$

$$\frac{3}{8} \checkmark$$

Question 5

a) $OP \perp OQ$

$$\therefore -\frac{1}{3}iZ = OQ$$



$$OR = OP + OQ$$

$$OR = Z - \frac{1}{3}iZ$$

$$OR = (1 - \frac{1}{3}i)Z$$

b) $\cos(ax - bx) = \cos ax \cos bx + \sin ax \sin bx$
i) $\cos(ax + bx) = \cos ax \cos bx - \sin ax \sin bx$

Subtract. \cos cancel
 \sin add

$$2 \sin ax \sin bx$$

ii) $\int \sin 3x \sin 2x dx = \frac{1}{2} \int (\sin 3x \sin 2x) dx$
 $= \frac{1}{2} \int (\cos(3-2)x - \cos(3+2)x) dx$
 $= \frac{1}{2} \int (\cos x - \cos 5x) dx$
 $= \frac{1}{2} \left[-\sin x - \frac{1}{5} \sin 5x \right]$
 $= \frac{\sin x}{2} - \frac{\sin 5x}{10} + C$

Question 4

c) i)
$$U_3 = 4U_2 - 4U_1 = 48$$
$$U_4 = 4U_3 - 4U_2 = 112$$

ii) Test for $n=1, n=2$

$$n=1 \quad U_1 = 4 \times 2 = 8 \quad \checkmark$$

$$n=2 \quad U_2 = 5 \times 4 = 20 \quad \checkmark$$

Assume true for k and $k+1$.
$$U_k = (k+3)2^k \quad U_{k+1} = (k+4)2^{k+1}$$

Prove $n=k+2 = (k+5)2^{k+2}$

$$\begin{aligned} U_{k+2} &= 4U_{k+1} - 4U_k \\ &= 4(k+4)2^{k+1} - 4(k+3)2^k \\ &= 4 \cdot 2^k \{ 2(k+4) - (k+3) \} \\ &= 2^2 \cdot 2^k \{ k+5 \} \\ &= 2^{k+2} (k+5) \end{aligned}$$

If true for $n=k, n=k+1$ and true for $n=k+2$

If true for $n=1, 2$.

\therefore By induction true for $n=3, \dots$

for all integer $n \geq 1$.

Question 4

d)

$$ax^3 + bx^2 + d = 0$$

Double root

$\therefore f'(x) = 0$ will be root

$$3ax^2 + 2bx = 0$$

$$x(3ax + 2b) = 0$$

Two options $x = 0$

$$x = \frac{-2b}{3a}$$

$$P(0) \neq 0 \quad \therefore P\left(\frac{-2b}{3a}\right) = 0$$

$$- \cancel{a} \times \frac{8b^3}{27a^3} + b \frac{4b^2}{9a^2} + d = 0$$

$$\times 27a^2 \quad -8b^3 + 12b^3 + 27a^2d = 0$$

$$27a^2d + 4b^3 = 0$$

Question 7

$$a) \int \frac{dx}{\sqrt{4-9x}}$$

$$\frac{1}{3} \int \frac{dx}{\sqrt{\frac{4}{9} - x^2}}$$

$$I = \frac{1}{3} \sin^{-1} \frac{3x}{2}$$

$$b) \int x e^x dx$$

$$u = x \quad dv = e^x$$
$$du = 1 \quad v = e^x$$

$$I = [x e^x - \int e^x dx]$$

$$I = x e^x - e^x$$

$$\int \cos^5 x \sin^3 x dx$$
$$\int \cos^3 x (1 - \cos^2 x) \sin x dx$$
$$\int \cos^5 x \sin x - \cos^7 x \sin x dx$$

$$I = \frac{1}{6} \cos^6 x - \frac{1}{8} \cos^8 x$$

$$c) \int_0^{\sqrt{2}} \frac{x^3}{x^2+4} dx$$

$$x^2+4 \quad \frac{x}{x^3+4x} \\ \frac{x^3}{-4x}$$

$$x - \frac{4x}{x^2+4}$$

$$I = \int_0^{\sqrt{2}} x - \frac{4x}{x^2+4} dx$$

$$= \left[\frac{x^2}{2} - 2 \ln(x^2+4) \right]_0^{\sqrt{2}}$$

$$= 1 - 2 \ln 6 + 2 \ln 4$$

$$= 1 + 2 \ln \frac{2}{3}$$

Question 6

$$d \quad \frac{2(x+1)}{(x-1)(2x-1)} = \frac{A}{(x-1)} + \frac{B}{(2x-1)}$$

$$2(x+1) = A(2x-1) + B(x-1)$$

$$x=1 \quad 4 = 3A \quad \therefore A = \frac{4}{3}$$

$$x = -\frac{1}{2} \quad 1 = -\frac{3}{2}B \quad B = -\frac{2}{3}$$

$$\int_2^5 \frac{2(x+1)}{(x-1)(2x-1)} = \int_2^5 \frac{\frac{4}{3}}{(x-1)} - \int_2^5 \frac{\frac{2}{3}}{(2x-1)}$$

$$= \left[\frac{4}{3} \ln(x-1) - \frac{2}{3} \ln(2x-1) \right]_2^5$$

$$= \frac{1}{3} \left[4 \ln 4 - \ln(11) - 4 \ln 1 + \ln 5 \right]$$

$$= \frac{1}{3} \left[\frac{\ln 256 \times 5}{11} \right]$$

Question 6

$$e) U_n = \int_0^{\frac{\pi}{2}} x^n \sin x$$

$$u = x^n \quad dv = \sin x$$

$$du = nx^{n-1} \quad v = -\cos x$$

$$U_n = \left[-x^n \cos x \right]_0^{\frac{\pi}{2}} + \int_0^{\frac{\pi}{2}} nx^{n-1} \cos x dx$$

$$u = nx^{n-1} \quad dv = \cos x$$

$$du = n(n-1)x^{n-2} \quad v = \sin x$$

$$U_n = \left[nx^{n-1} \sin x \right]_0^{\frac{\pi}{2}} + \int_0^{\frac{\pi}{2}} n(n-1)x^{n-2} \sin x dx$$

$$\downarrow$$
$$U_{n-2}$$

$$U_n + U_{n-2} = n \left(\frac{\pi}{2} \right)^{n-1} \quad n \geq 2$$

$$\begin{aligned} \int_0^{\frac{\pi}{2}} x^4 \sin x &= 4 \left(\frac{\pi}{2} \right)^3 - \int_0^{\frac{\pi}{2}} x^2 \sin x \\ &= 4 \left(\frac{\pi}{2} \right)^3 - 2 \left(\frac{\pi}{2} \right) + \int_0^{\frac{\pi}{2}} \sin x dx \\ &= \frac{\pi}{2} - \pi + \left[-\cos x \right]_0^{\frac{\pi}{2}} \end{aligned}$$

$$I = 1 - \frac{\pi}{2}$$

Question 8

$$\sqrt{1} + \sqrt{2} + \sqrt{3} + \dots + \sqrt{n} \leq \frac{4n+3}{6} \sqrt{n} \quad n \geq 1$$

Step 1 $n \geq 1$ $\sqrt{1} \leq \frac{7}{6} \sqrt{1}$ ✓

Step 2 $\sqrt{1} + \sqrt{2} + \dots + \sqrt{k} \leq \frac{4k+3}{6} \sqrt{k}$

Prove $\sqrt{1} + \sqrt{2} + \dots + \sqrt{k} + \sqrt{k+1} \leq \frac{4(k+1)+3}{6} \sqrt{k+1}$

$$\frac{4k+3}{6} \sqrt{k} + \sqrt{k+1} \leq \frac{4k+7}{6} \sqrt{k+1}$$

$$\frac{4k+7}{6} \sqrt{k+1} - \frac{4k+3}{6} \sqrt{k} - \sqrt{k+1} < 0$$

$$\frac{4k+1}{6} \sqrt{k+1} - \frac{4k+3}{6} \sqrt{k} < 0$$

Since $k \geq 1$ $\sqrt{k+1} > \sqrt{k}$.

$$\frac{4k+1}{6} \sqrt{k+1} - \frac{4k+3}{6} \sqrt{k+1}$$

$$-\frac{1}{3} \sqrt{k+1}$$

Since $\sqrt{k+1} > 0$

$$-\frac{1}{3} \sqrt{k+1} < 0$$

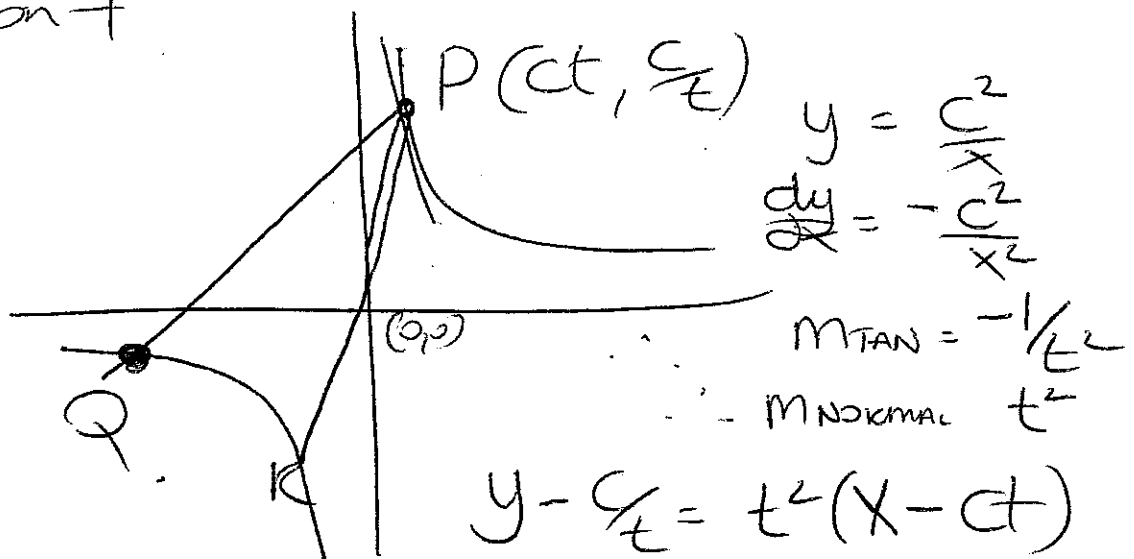
Statement is true.

$$n = 1, 2$$

True for $k = 1$

By Mathematical Induction

Question 7



$$y = \frac{c^2}{x}$$

$$\frac{dy}{dx} = -\frac{c^2}{x^2}$$

$$m_{TAN} = -\frac{1}{t^2}$$

$$m_{NORMAL} = t^2$$

$$y - \frac{c}{t} = t^2(x - ct)$$

$$yt - c = t^3x - ct^4$$

$$t^3x - ty = ct^4 - c$$

ii) Sub $y = \frac{c^2}{x}$ into $t^3x - ty = ct^4 - c$

$$t^3x - \frac{tc^2}{x} = ct^4 - c$$

$$t^3x^2 - (ct^4 - c)x - tc^2 = 0$$

Product of roots $\frac{tc^2}{t^3} = ct \times \text{root}$

$$P_x = -\frac{c}{t^3} \quad P_y = ct^3 \quad \left[-\frac{c}{t^3}, -ct^3 \right]$$

iii) By symmetry $R = (-ct, -c/t)$

$$m_{QR} = \frac{-ct^3 - (-c/t)}{-c/t^3 - (-ct)}$$

$$= \frac{\frac{c - ct^4}{t}}{\frac{ct^4 - c}{t^3}}$$

$$= \frac{-1}{t^2}$$

$$m_{PR} = \frac{\frac{c}{t} - (-c/t)}{ct - (-ct)}$$

$$= \frac{\frac{2c}{t}}{2ct}$$

$$= \frac{1}{t^2}$$

⊥ (almost)!

$$P(c/t, c/t)$$

$$Q(-c/t^3, -ct^3)$$

$$\left(\frac{ct - c/t^3}{2}, \frac{c/t - ct^3}{2} \right)$$

$$\left(\frac{ct^4 - c}{2t^3}, \frac{c - ct^4}{2t} \right)$$

$$\left(\frac{ct^4 - c}{2t^3}, -\frac{(ct^4 - c)}{2t} \right)$$

x

y

$$xy = t^2$$

Rectangula Hyperbola