The Scots College

HSC Mathematics Extension 2

Trial

August 2010

Name:______________________

General Instructions

- Working time : 3 hours + 5 minutes Reading time.
- Write using blue or black pen
- Board approved calculators may be Used (Non Graphic)
- All necessary working should be shown in every question
- Standard Integrals Table attached

TOTAL MARKS: 120

Attempt Questions 1 - 8
All questions are of equal value

WEIGHTING: 40 %

Answer each question in a SEPARATE writing booklet.

Question 1 (Marks 15) Use a SEPARATE writing booklet.

a) Evaluate [2]

$$\int_0^1 \frac{2\,dx}{\sqrt{2 - x^2}}$$

b) Find [2]

$$\int \frac{\tan x}{\ln(\cos x)}\,dx$$

c) Evaluate [3]

$$\int_0^1 sin^{-1}x\,dx$$

d) [4]

i) Find the real numbers $a,\ b,\ and\ c$ such that

$$\frac{1 + 4x}{(x^2 + 1)(4 - x)} \equiv \frac{a}{4 - x} + \frac{bx + c}{x^2 + 1}$$

ii) Hence evaluate

$$\int_0^2 \frac{1+4x}{(x^2+1)(4-x)}\,dx$$

e) By using the substitution $x = 2\cos\theta$ or otherwise evaluate [4]

$$\int_{-2}^0 \sqrt{4 - x^2}\,dx$$

Question 2 (Marks 15) Use a SEPARATE writing booklet.

a) Express $\left(\frac{1+i}{1-i}\right)^3$ in the form $a + ib$; where a and b are real numbers [2]

b) i) Write $\frac{5 - i}{2 - 3i}$ in the form $x + iy$, where x and y are real numbers. [2]

ii) Write $\frac{5 - i}{2 - 3i}$ in the modulus-argument form. [3]

iii) Calculate $\left(\frac{5 - i}{2 - 3i}\right)^4$. [1]

c) Sketch the region on the Argand diagram where the inequalities [3]

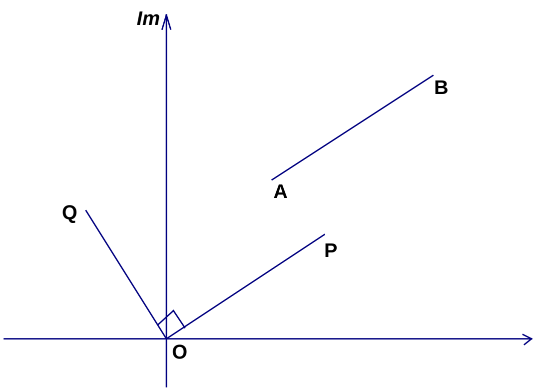
$$|z - \bar{z}| < 4 \quad \text{and} \quad |z + 1| > 1$$

hold simultaneously.

d) In the Argand diagram below, $AB = OP = OQ$, $OP \parallel AB$ and $OP \perp OQ$. [4]
A represents the complex number $5 + 3i$ and B represents $11 + 5i$. Copy this diagram into your answer scripts and find the complex numbers represented by the point :

i) P

ii) Q



Question 3 (Marks 15) Use a SEPARATE writing booklet.

a) i) Determine whether $f\ (x) = \frac{x}{x^2 - 1}$ is an odd or even function. [6]

ii) Sketch the graph of $y = f(x)$.

iii) Using the graph of $y = f(x)$, sketch on separate axes, the graphs of

(α) $y = f(-x)$

(β) $y = |f(x)|$

(γ) $y = [f(x)]^2$

b) Tangents to the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b} = 1$ at points $P(x_1, y_1)$ and $Q(x_2, y_2)$ intersect at T. M is the mid-point of PQ. [6]

i) Given that the tangent to the ellipse at P has equation $\frac{xx_1}{a^2} + \frac{yy_1}{b^2} = 1$ write down the equation of the tangent to the ellipse at Q.

ii) Show that the line $\frac{xx_1}{a^2} + \frac{yy_1}{b^2} = \frac{xx_2}{a^2} + \frac{yy_2}{b^2}$ passes through T and M.

iii) Deduce that $O, T\ and\ M$ are collinear.

iv) If $\angle PTQ$ is a right angle, show that $\frac{x_1x_2}{a^4} + \frac{y_1y_2}{b^4} = 0$

c) i) If $a > b > 0$, sketch the curve and shade the region $\int_b^a \sqrt{a^2 - x^2}\ dx$. [3]

ii) By using your diagram, or otherwise, show that

$$\int_b^a \sqrt{a^2 - x^2}\ dx = \frac{a^2}{2}\ cos^{-1}\left(\frac{b}{a}\right) - \frac{b}{a}\ \sqrt{a^2 - x^2}$$

Question 4 (Marks 15) Use a SEPARATE writing booklet.

a) i) Sketch the graph of the curve $y = x + e^{-x}$ showing clearly the coordinates of any turning points and the equations of any asymptotes. [2]

ii) The region in the first quadrant between the curve $y = x + e^{-x}$ and the line $y = x$ and bounded by the line $x = 1$ is rotated through one complete revolution about the y-axis. Use the method of cylindrical shells to show that the volume V of the solid of revolution is given by [3]

$$V = 2\pi \int_0^1 xe^{-x}\ dx$$

iii) Find the volume of the solid of revolution. [3]

b) Let $f(x) = x^2(x^2 - 2)$. The tangent to the curve $y = f(x)$ at the point A with x coordinate α meets the curve again at B. [7]

i) Show the tangent AB has equation $y = 4\alpha(\alpha^2 - 1)x + \alpha^2(2 - 3\alpha^2)$.

ii) Deduce that $x^2(x^2 - 2) = 4\alpha(\alpha^2 - 1)x + \alpha^2(2 - 3\alpha^2)$ has real roots $\alpha, \alpha, \beta, \gamma$ for some β, γ.

iii) For $\alpha \neq 0$, find $\beta + \gamma$ and $\beta\gamma$ in terms of α and write down a quadratic equation with roots β, γ.

iv) Find the possible values of α.

Question 5 (Marks 15) Use a SEPARATE writing booklet.

a) If $u_1 = 12,\ u_2 = 30$ and $u_n = 5u_{n-1} - 6u_{n-2}$ for $n \geq 3$. [5]

(i) Determine u_3 and u_4 .

(ii) Show that $u_n = 2 \times 3^n + 3 \times 2^n$ for $n = 1$ **and** $n = 2$.

(iii) If $u_k = 2 \times 3^k + 3 \times 2^k$ and $u_{k+1} = 2 \times 3^{k+1} + 3 \times 2^{k+1}$, where k is a positive integer, prove that

$$u_{k+2} = 2 \times 3^{k+2} + 3 \times 2^{k+2}.$$

b) (i) Show that $\operatorname{cosec} 2\theta + \cot 2\theta = \cot\theta$ for all real values of θ . [7]

(ii) Use the result above to :

(α) find in surd form the values of $\cot\frac{\pi}{8}$ and $\cot\frac{\pi}{12}$.

(β) show without using calculators that

$$\operatorname{cosec}\frac{4\pi}{15} + \operatorname{cosec}\frac{8\pi}{15} + \operatorname{cosec}\frac{16\pi}{15} + \operatorname{cosec}\frac{32\pi}{15} = 0$$

c) i) Show that for $a > 0$ and $n \neq 0$, [3]

$$\log_{a^n}(x) = \frac{1}{n}\log_a x \ .$$

ii) Hence evaluate

$$\log_2 3 + \log_4 3 + \log_{16} 3 + \log_{256} 3 + \cdots \ .$$

Question 6 (Marks 15) Use a SEPARATE writing booklet.

a) The base of a certain solid is the region between the x-axis and the curve $y = \sin 2x$ between $x = 0$ and $= \frac{3\pi}{8}$. [6]

Each plane section of the solid perpendicular to the x-axis is an equilateral triangle with one side in the base of the solid.

Find the volume of the solid.

b) In the expansion of $(ax - bx^{-2})^8$ the coefficient of x^2 and x^{-1} are equal. [4]

Show that $a + 2b = 0$.

c) A particle of mass m kg falls from rest in a medium where the resistance to motion is mkv when the particle has velocity $v\ ms^{-1}$. [5]

i) Draw a diagram showing the forces acting on the particle.

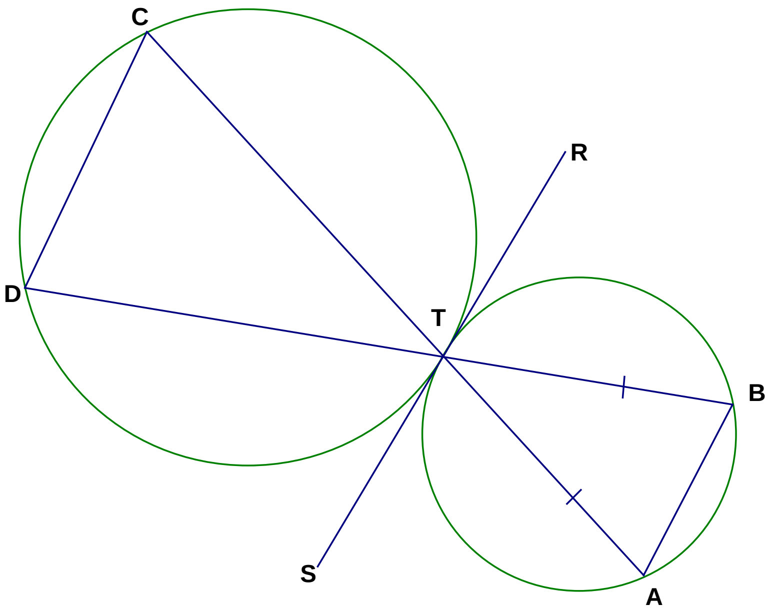
ii) Show that the equation of motion of the particle is $\ddot{x} = k(V - v)$ where $V\ ms^{-1}$ is the terminal velocity of the particle in this medium, and x metres is the distance fallen in t seconds.

iii) Find the time T seconds taken for the particle to attain 50% of its terminal **velocity**, and the **distance** fallen in time in terms of V and k .

Question 7 (Marks 15) Use a SEPARATE writing booklet.

a) Two circles touch externally at a point T. [8]

A and B are points on the first circle such that $AT = BT$, and AT and BT produced meet the second circle at C and D respectively. RS is the common tangent at T. Let $\angle BAT = \alpha$.



i) Copy the diagram and include the information above.

ii) Prove that $\angle BAC = \angle ACD$.

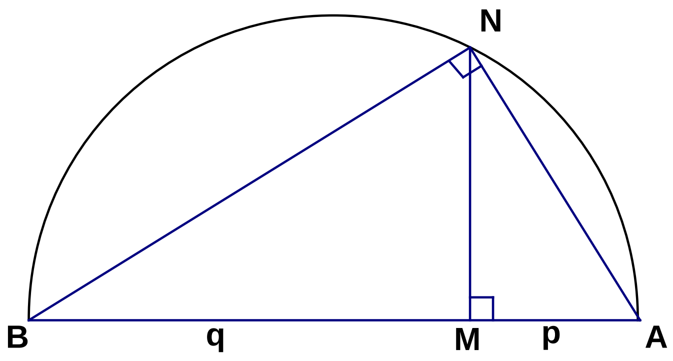
iii) Prove that $ABCD$ is a trapezium with two equal sides.

The line BC cuts the first circle in V and the second circle again in , and the line AD cuts the first circle in U and the second circle again in X.

iv) Prove that the points U, V, W and X are concyclic.

Question 7 continued...........

b) In the diagram, AB is the diameter of a semicircle. The angle ANB is $90°$ and M is a point on AB such that NM is perpendicular to AB. [7]



If $AM = p$ and $MB = q$,

i) Show that $NM = \sqrt{pq}$.

ii) Deduce, using the diagram, that $\sqrt{pq} \leq \frac{p+q}{2}$.

iii) Use (ii) to prove that if $p, q, x, y > 0$, then

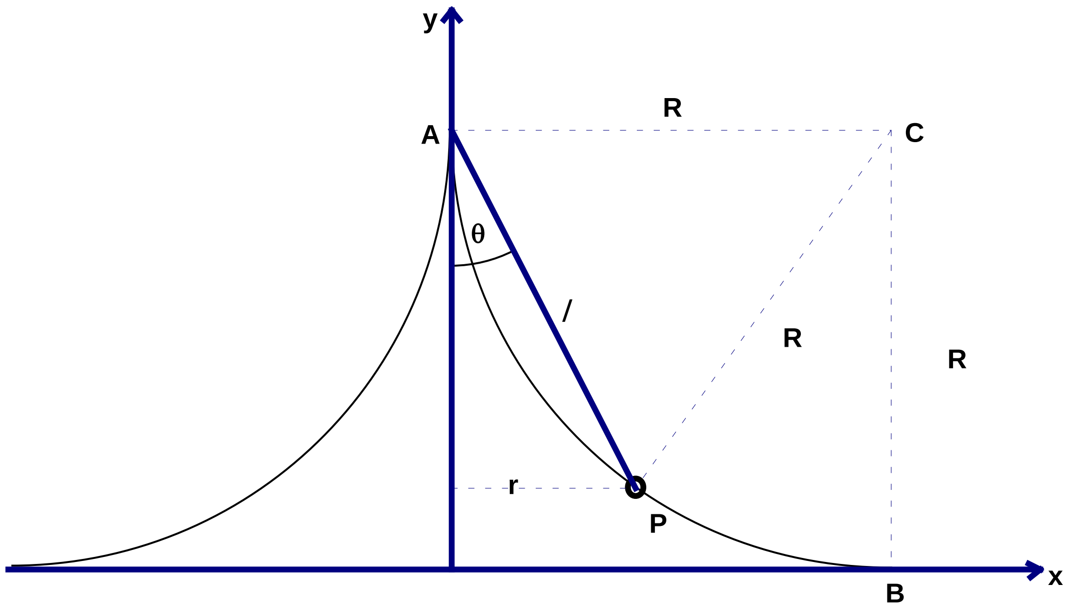
$$\frac{1}{4}(p + q + x + y) \geq (pqxy)^{\frac{1}{4}}$$

iv) Deduce that if $l, m, n, z > 0$, then $\frac{l}{m} + \frac{m}{n} + \frac{n}{z} + \frac{z}{l} \geq 4$

End of Question 7

Question 8 (Marks 15) Use a SEPARATE writing booklet.

a) [10]



AB is an arc of a circle centre C and radius R . A surface is formed by rotating the arc AB through one revolution about the y-axis. A light, inextensible string of length l , $l \leq R$, is attached to point A , and a particle of mass m is attached to the other end. The particle is set in motion, tracing out a horizontal circle on the surface with constant angular velocity ω radians per second, while the string stays taught.

i) Explain why, when the particle is in position P shown on the diagram, the direction of the force N exerted by the surface on the particle is towards C .

ii) If the string makes an angle θ with the vertical, show that $\angle ACP = 2\theta$.

iii) Show on a diagram the tension force T , the force N and the weight force of magnitude mg acting on the particle, indicating their direction in terms of θ .

iv) Show that

$$T \cos\theta + N \sin 2\theta = mg$$

$$T \sin\theta - N \cos 2\theta = m\, l \sin\theta\, \omega^2$$

v) Show that

$$N = m\, l \sin\theta \left(\frac{g}{l} \sec\theta - \omega^2\right) .$$

vi) Deduce that there is a maximum value ω for the motion to occur as described, and write down this maximum value.

Question 8 continued........

b) i) Show that $\tan^{-1}(n+1) - \tan^{-1}(n-1) = \tan^{-1}\frac{2}{n^2}$, where n is a positive integer. [5]

ii) Hence or otherwise show that for $n \geq 1$

$$\sum_{r=1}^{n} \tan^{-1}\frac{2}{r^2} = \frac{3\pi}{4} + \tan^{-1}\frac{2n+1}{1-n-n^2}$$

End of Assessment

Standard Integrals

$\int x^n\,dx$	$= \dfrac{1}{n+1}x^{n+1}, \quad n \neq -1; x \neq 0, \text{if } n < 0$
$\int \dfrac{1}{x}\,dx$	$= \ln x, \quad x > 0$
$\int e^{ax}\,dx$	$= \dfrac{1}{a}e^{ax}, \quad a \neq 0$
$\int \cos ax\,dx$	$= \dfrac{1}{a}\sin ax, \quad a \neq 0$
$\int \sin ax\,dx$	$= -\dfrac{1}{a}\cos ax, \quad a \neq 0$
$\int \sec^2 ax\,dx$	$= \dfrac{1}{a}\tan ax, \quad a \neq 0$
$\int \sec ax \tan ax\,dx$	$= \dfrac{1}{a}\sec ax, \quad a \neq 0$
$\int \dfrac{1}{a^2+x^2}\,dx$	$= \dfrac{1}{a}\tan^{-1}\dfrac{x}{a}, \quad a \neq 0$
$\int \dfrac{1}{\sqrt{a^2-x^2}}\,dx$	$= \sin^{-1}\dfrac{x}{a}, \quad a > 0, \quad -a < x < a$
$\int \dfrac{1}{\sqrt{x^2-a^2}}\,dx$	$= \ln\left(x+\sqrt{x^2-a^2}\right), \quad x > a > 0$
$\int \dfrac{1}{\sqrt{x^2+a^2}}\,dx$	$= \ln\left(x+\sqrt{x^2+a^2}\right)$

NOTE : $\ln x = \log_e x, \quad x > 0$