Sydney Girls High School

12MA1

2010

TRIAL HIGHER SCHOOL CERTIFICATE EXAMINATION

# Mathematics Extension 2

**General Instructions**

- Reading Time - 5 minutes
- Working time - 3 hours
- Attempt ALL questions
- ALL questions are of equal value
- All necessary working should be shown in every question. Marks may be deducted for careless or badly arranged work.
- Standard integrals are supplied
- Board-approved calculators may be used.
- Diagrams are not to scale
- Each question attempted should be started on a new page. Write on one side of the paper only.

This is a trial paper ONLY. It does not necessarily reflect the format or the content of the 2010 HSC Examination Paper in this subject.

| 2 | 0 | 0 | 5 | 9 | 9 | 1 | 5 | |
|---|---|---|---|---|---|---|---|---|

Candidate Number

STANDARD INTEGRALS

$$\int x^n\,dx = \frac{1}{n+1}x^{n+1}, \quad n \neq -1; \quad x \neq 0, \text{ if } n < 0$$

$$\int \frac{1}{x}\,dx = \ln x, \quad x > 0$$

$$\int e^{ax}\,dx = \frac{1}{a}e^{ax}, \quad a \neq 0$$

$$\int \cos ax\,dx = \frac{1}{a}\sin ax, \quad a \neq 0$$

$$\int \sin ax\,dx = -\frac{1}{a}\cos ax, \quad a \neq 0$$

$$\int \sec^2 ax\,dx = \frac{1}{a}\tan ax, \quad a \neq 0$$

$$\int \sec ax \tan ax\,dx = \frac{1}{a}\sec ax, \quad a \neq 0$$

$$\int \frac{1}{a^2+x^2}\,dx = \frac{1}{a}\tan^{-1}\frac{x}{a}, \quad a \neq 0$$

$$\int \frac{1}{\sqrt{a^2-x^2}}\,dx = \sin^{-1}\frac{x}{a}, \quad a > 0, \quad -a < x < a$$

$$\int \frac{1}{\sqrt{x^2-a^2}}\,dx = \ln\left(x+\sqrt{x^2-a^2}\right), \quad x > a > 0$$

$$\int \frac{1}{\sqrt{x^2+a^2}}\,dx = \ln\left(x+\sqrt{x^2+a^2}\right)$$

NOTE : $\ln x = \log_e x, \quad x > 0$

## Question One (15 marks)

Marks

a) Find $\int \frac{\sin x}{\cos^3 x}\,dx$. 2

b) Find $\int \frac{4x^3 - 2x^2 + 1}{2x - 1}\,dx$. 3

c) i) Find $A$, $B$ and $C$ given that $\frac{4x-6}{(x+1)(2x^2+3)} = \frac{A}{x+1} + \frac{Bx+C}{2x^2+3}$. 3

ii) Hence, find $\int \frac{4x-6}{(x+1)(2x^2+3)}\,dx$. 1

d) Find $\int \sin^{-1} x\,dx$. 3

e) Find $\int \frac{1}{3+2\cos\theta}\,d\theta$. 3



## Question Two (15 marks)

Marks

a) If $z = \sqrt{3} - i$ and $w = 1 + i$, find :

i) $zw$ 1

ii) $\arg z$ 1

iii) $|w^7|$ 1

iv) $\text{Im}\left(\frac{z}{w}\right)$ 2

b) $OPQR$ is a rectangle on the Argand diagram labelled anti-clockwise where O represents the origin and point $P$ represents the complex number $3 + 4i$. Find the complex number representing $Q$ and $R$ given that $PQ = 2QR$. 2

c) i) Find the square roots of $21 + 20i$. 2

ii) Hence, solve $(1+i)z^2 + z - 5 = 0$. 3

d) The complex number $z$ is such that $|z-1| = \text{Re}(z)$.

i) Find the cartesian equation of the locus of $z$. 2

ii) Find the range of values of $|z|$. 1

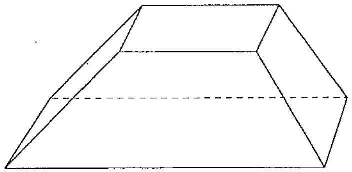
## Question Three (15 marks) Marks

a) The region bounded by $y=\log_e x$ , $y=1$ and $x=3$ is rotated about the $y$ axis.

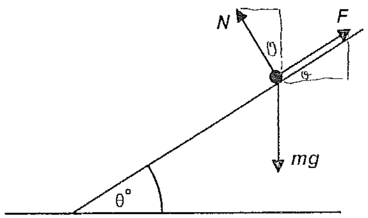
i) Sketch this region on the number plane. 1

ii) Find the volume formed using the method of **cylindrical shells**. 3

b) A solid is formed with the base and top both rectangles parallel to each other and 6 cm apart. The dimensions of the base are 11 cm and 15 cm and the dimensions of the top are 7 cm and 10 cm. If all other faces are trapeziums, find the volume of the solid. 5



c) An object of mass $m$ is lying on an inclined plane at an angle $\theta$ to the horizontal. As shown in the diagram below, the object is subject to a gravitational force $mg$, a normal reaction force $N$ and a frictional force $F$.



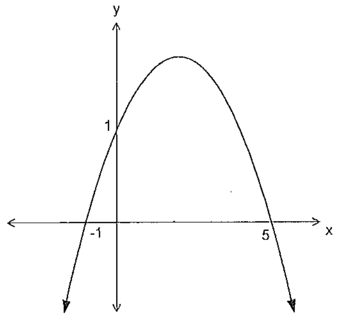

The object is **not** moving.
Resolve the forces acting on the object , and hence find an expression for $\frac{F}{N}$ in terms of $\theta$. 3

d) Find $\frac{dy}{dx}$ given $x^3+x^2y^4=0$. 3



## Question Four (15 marks) Marks

a) The diagram shows the graph of $y=f(x)$.




Draw separate one-third page sketches of the graphs of the following :

i) $y=|f(x)|$ 1

ii) $y=f(|x|)$ 1

iii) $y=[f(x)]^2$ 2

iv) $y=e^{f(x)}$ 2

v) $y=-\frac{1}{f(x)}$ 3

b) The equation $x^3+3x^2-2x-2=0$ has roots $\alpha$, $\beta$ and $\gamma$.
Find the equation with roots $\frac{2\alpha}{\beta\gamma}$, $\frac{2\beta}{\alpha\gamma}$ and $\frac{2\gamma}{\alpha\beta}$. 3

c) Determine the greatest and least values of $\arg(z)$ if $|z-4i|=2$. 3

## Question Five (15 marks)

Marks

a) Given $1-i$ is a root of $x^3-3x^2+4x-2=0$ find the other roots. 2

b) For the equation $x^4+x^3-16x^2-4x+48=0$, the product of two of the roots is 6.

i) Hence, express the equation in the form $(x^2+ax+b)(x^2+cx+d)=0$. 3

ii) Find the roots of the equation. 1

c) i) Given $x=\alpha$ is a double root of the equation $ax^4+4bx+c=0$, deduce that $\alpha^3=-\frac{b}{a}$. 2

ii) Also, deduce that $ac^3=27b^4$. 3

iii) Hence or otherwise, solve the equation $27x^4-32x+16=0$, given that it has a double root. 4



## Question Six (15 marks)

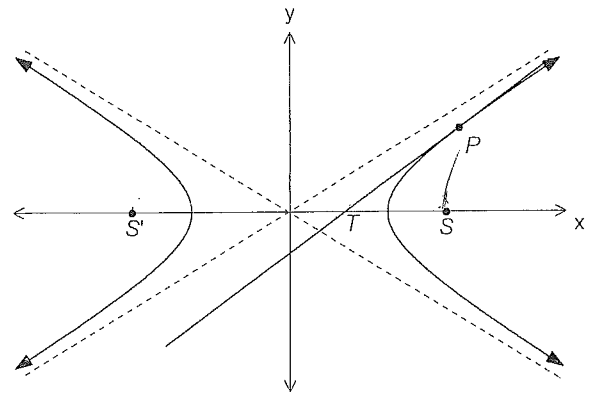
Marks

a) $P\left(p,\frac{1}{p}\right)$ and $Q\left(q,\frac{1}{q}\right)$ are two points on the rectangular hyperbola $xy=1$.

i) Derive the equation of the chord $PQ$ and show that it can be expressed in general form as $x+pqy-(p+q)=0$. 2

ii) Hence, show that the area of $\Delta OPQ$ is $\frac{|p^2-q^2|}{2|pq|}$ units$^2$. 4

b) The point $P(x_1,y_1)$ lies on the hyperbola $\frac{x^2}{16}-\frac{y^2}{9}=1$. The tangent at $P$ cuts the $x$-axis at $T$.




i) Find the coordinates of the foci $S$ and $S'$. 1

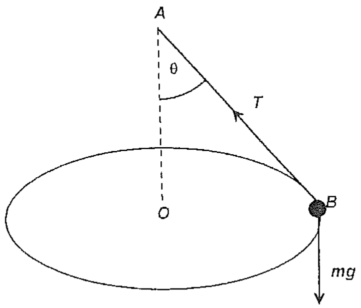
ii) Show that the equation of the tangent at $P$ is given by $\frac{xx_1}{16}-\frac{yy_1}{9}=1$. 3

iii) Show that $\frac{S'T}{ST}=\frac{S'P}{SP}$. 3

iv) Hence, deduce that $\angle S'PT=\angle SPT$. 2

## Question Seven (15 marks) — Marks

a) A particle of mass $m$ kg is attached to one end of a light string at $B$. The other end of the string is fixed at a point $A$. The particle rotates in a horizontal circle of radius $r$ metres at $g$ rad/s, the centre of the circle being directly below $A$.




The forces acting on the particle are the tension in the string $T$ and the gravitational force $mg$.

Let $\angle BAO = \theta$.

i) Show that $T\sin\theta = mg^2 r$. 1

ii) Prove that $\theta = \tan^{-1}(gr)$. 1

iii) Prove that $T = mg\sqrt{1+g^2r^2}$. 2

b) i) Use De Moivre's Theorem to express $\cos 4\theta$ and $\sin 4\theta$ as powers of $\cos\theta$ and $\sin\theta$. 2

ii) Hence show that $\tan 4\theta = \dfrac{4t-4t^3}{1-6t^2+t^4}$ where $t=\tan\theta$. 1

iii) By first solving the equation $\tan 4\theta = 1$ for $0 \le \theta \le 2\pi$, solve the equation $x^4+4x^3-6x^2-4x+1=0$. 3

iv) Hence find the value of $\tan\frac{\pi}{16}\tan\frac{3\pi}{16}\tan\frac{5\pi}{16}\tan\frac{7\pi}{16}$. 2

c) Evaluate $\displaystyle\int_0^{\pi} x\cos 2x\, dx$. 3



## Question Eight (15 marks) — Marks

a) i) Use integration by parts to show that $I_n = \dfrac{n-1}{n} I_{n-2}$ for $n \ge 2$ given

$I_n = \displaystyle\int_0^{\frac{\pi}{2}} (\sin x)^n\, dx$ where $n$ is an integer ($n \ge 0$). 3

ii) Deduce $I_{2n} = \dfrac{2n-1}{2n}\times\dfrac{2n-3}{2n-2}\times\ldots\times\dfrac{3}{4}\times\dfrac{1}{2}\times\dfrac{\pi}{2}$. 3

b) Given the identity $\cos(A+B)+\cos(A-B) = 2\cos A\cos B$, solve the equation $\cos 5x + \cos 3x + 2\cos x = 0$ for $0 \le x \le \frac{\pi}{2}$. 3

c) Two sides of a triangle are of length $2x$ cm and $3x$ cm. The angles opposite these sides differ by $45°$. Show that the smaller of the two angles is given by $\tan^{-1}\left(\dfrac{2+3\sqrt{2}}{7}\right)$. 4

d) The positive integers are bracketed as follows $(1), (2,3), (4,5,6), (7,8,9,10), \ldots\ldots$.
The $n$th bracket has $n$ integers.
Prove that the sum of the integers in the $n$th bracket is $\frac{n}{2}(n^2+1)$. 2

**End of paper**

BLANK PAGE

QUESTION 1

a. Let $u=\cos x$

$$\frac{du}{dx}=-\sin x$$

$$\int\frac{\sin x}{\cos^3 x}dx=\int -u^{-3}du$$
$$=-\frac{u^{-2}}{-2}+C$$
$$=\frac{1}{2\cos^2 x}+C$$

b.

$$2x-1\overline{)4x^3-2x^2+1}\quad 2x^2$$
$$4x^3-2x^2$$
$$0+1$$

$$\int\frac{4x^3-2x^2+1}{2x-1}dx=\int\left(2x^2+\frac{1}{2x-1}\right)dx$$
$$=\frac{2x^3}{3}+\frac{1}{2}\ln(2x-1)+C$$

c.

i. $4x-6=A(2x^2+3)+(Bx+C)(x+1)$

let $x=-1$

$-10=5A$

$\therefore A=-2$

$0=2A+B$

$\therefore B=4$

$-6=3A+C$

$\therefore C=0$

ii.

$$\int\frac{4x-6}{(x+1)(2x^2+3)}dx=\int\frac{-2}{x+1}+\frac{4x}{2x^2+3}$$
$$=-2\ln(x+1)+\ln(2x^2+3)+C$$

d.

let

$u=\sin^{-1}x \qquad dv=dx$

$u'=\frac{1}{\sqrt{1-x^2}} \qquad v=x$

$$\int\sin^{-1}xdx=x\sin^{-1}x-\int\frac{x}{\sqrt{1-x^2}}dx$$

Let

$u=1-x^2$

$du=-2xdx$

$-\frac{1}{2}du=xdx$

$$x\sin^{-1}x-\int\frac{x}{\sqrt{1-x^2}}dx=x\sin^{-1}x-\int\frac{-du}{2\sqrt{u}}$$
$$=x\sin^{-1}x+\frac{1}{2}\int u^{-\frac{1}{2}}$$
$$=x\sin^{-1}x+\frac{1}{2}\times 2u^{\frac{1}{2}}$$
$$=x\sin^{-1}x+u^{\frac{1}{2}}$$
$$=x\sin x+\sqrt{1-x^2}$$

e.

$$\int\frac{1}{3+2\cos\theta}d\theta=\int\frac{1}{3+2\left(\frac{1-t^2}{1+t^2}\right)}\bullet\frac{2dt}{1+t^2}$$
$$=\int\frac{1}{\frac{5+t^2}{1+t^2}}\bullet\frac{2dt}{1+t^2}$$
$$=\int\frac{2}{5+t^2}dt$$
$$=2\left(\frac{1}{\sqrt{5}}\tan^{-1}\frac{t}{\sqrt{5}}\right)+C$$
$$=2\left(\frac{1}{\sqrt{5}}\tan^{-1}\frac{\tan\frac{\theta}{2}}{\sqrt{5}}\right)+C$$

QUESTION 2

a.

i.

$$zw=(\sqrt{3}-i)(1+i)$$
$$=\sqrt{3}-i^2+i\left(\sqrt{3}-1\right)$$
$$=\sqrt{3}+1+i\left(\sqrt{3}-1\right)$$

ii.

$$\arg(z)=-\tan^{-1}\left(\frac{1}{\sqrt{3}}\right)$$
$$=-\frac{\pi}{6}$$

iii.

$$|w|=\sqrt{1^2+1^2}$$
$$=\sqrt{2}$$
$$|w^7|=|w|^7$$
$$=\left(\sqrt{2}\right)^7$$
$$=8\sqrt{2}$$

iv.

$$\operatorname{Im}\left(\frac{z}{w}\right)=\operatorname{Im}\left(\frac{\sqrt{3}-i}{1+i}\times\frac{1-i}{1-i}\right)$$
$$=-\frac{\sqrt{3}+1}{2}$$

b.

$$R=2i(3+4i)$$
$$=-8+6i$$

$$Q=\overrightarrow{OR}+\overrightarrow{OP}$$
$$=-8+6i+3+4i$$
$$=-5+10i$$

c.

i.

$$\sqrt{21+20i}=a+ib$$
$$21+20i=a^2+2abi-b^2$$
$$a^2-b^2=21$$
$$2ab=20$$
$$a=\pm5$$
$$b=\pm2$$
$$\therefore\pm(5+2i)$$

ii.

$$z=\frac{-1\pm\sqrt{1^2+20(1+i)}}{2(1+i)}$$
$$=\frac{-1\pm(5+2i)}{2(1+i)}$$
$$=\frac{4+2i}{2(1+i)}\quad\text{or}\quad\frac{-6-2i}{2(1+i)}$$
$$=\frac{2+i}{1+i}\quad\text{or}\quad\frac{-3-i}{1+i}$$

d.

i.

$|z-1|=\operatorname{Re}(z)$

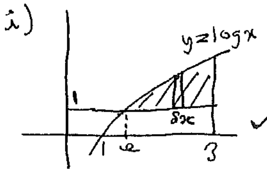
If $z=x+iy$

$$|(x-1)+iy|=x$$
$$\sqrt{(x-1)^2+y^2}=x$$
$$(x-1)^2+y^2=x^2$$
$$x^2-2x+1+y^2=x^2$$
$$y^2=2x-1\quad\text{or}\quad x=\frac{1}{2}\left(y^2+1\right)$$

ii.

$$|z|\geq\frac{1}{2}$$

3 a) i)




ii) $V_{shell} = \pi(R^2 - r^2)$

$= \pi\left((x+\delta x)^2 - x^2\right)(y-1)$

$= 2\pi x(y-1)\delta x$

$V_{solid} = \lim_{\delta x \to 0} \sum_{x=e}^{3} 2\pi x(y-1)\delta x$

$= 2\pi \int_e^3 x(y-1)\,dx$

$= 2\pi \int_e^3 (x\log x - x)\,dx$

$= 2\pi \int_e^3 x\log x - 2\pi\left[\frac{x^2}{2}\right]_e^3$

let $u = \log x$ $v' = x$

$u' = \frac{1}{x}$ $v = \frac{x^2}{2}$

$V_{solid} = 2\pi\left(\left[\log x \cdot \frac{x^2}{2}\right]_e^3 - \frac{1}{2}\int_e^3 x - \left[\frac{x^2}{2}\right]_e^3\right)$

$= 2\pi\left(\frac{9\log 3}{2} - \frac{3}{2}\left[\frac{x^2}{2}\right]_e^3 - \frac{e^2}{2}\right)$

$= 2\pi\left(\frac{9\log 3}{2} - \frac{27}{4} + \frac{e^2}{4}\right)$

$= \pi\left(9\log 3 - \frac{27}{2} + \frac{e^2}{2}\right) u^3$

b) $V_{slice} = Ah$

$= xy\,\delta h$

$V_{solid} = \lim_{\delta h \to 0} \sum_{h=0}^{6} xy\,\delta h$

$= \int_0^6 xy\,dh$

$h=0$ $x=7$ $y=10$

$h=6$ $x=11$ $y=15$

$x = mh+b$ $y = mh+b$

$7 = b$ $10 = b$

$11 = 6m+7$ $15 = 6m+10$

$4 = 6m$ $5 = 6m$

$m = \frac{2}{3}$ $m = \frac{5}{6}$

$= \int_0^6 \left(\frac{2h}{3}+7\right)\left(\frac{5h}{6}+10\right)dh$

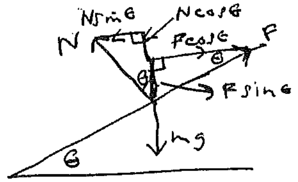
$= \int_0^6 \left(\frac{5h^2}{9} + \frac{75h}{6} + 70\right)dh$

$= \left[\frac{5h^3}{27} + \frac{25h^2}{4} + 70h\right]_0^6$

$= \frac{5\times 6^3}{27} + \frac{25\times 6^2}{4} + 70\times 6$

$= 685\,cm^3$

c)




$F\sin\theta + N\cos\theta - mg = 0$

$N\sin\theta - F\cos\theta = 0$

$F\sin^2\theta + N\sin\theta\cos\theta = mg\sin\theta$

$N\sin\theta\cos\theta - F\cos^2\theta = 0$

$\therefore F = mg\sin\theta$

$F\sin\theta\cos\theta + N\cos^2\theta = mg\cos\theta$

$N\sin^2\theta - F\sin\theta\cos\theta = 0$

$\therefore N = mg\cos\theta$

$\frac{F}{N} = \frac{mg\sin\theta}{mg\cos\theta}$

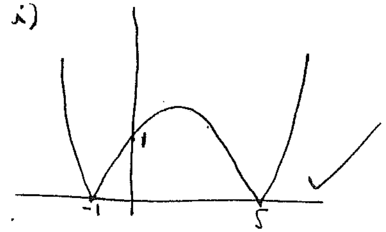
$= \tan\theta$

d) $3x^2 + x^2 \times 4y^3\frac{dy}{dx} + 2xy^4 = 0$

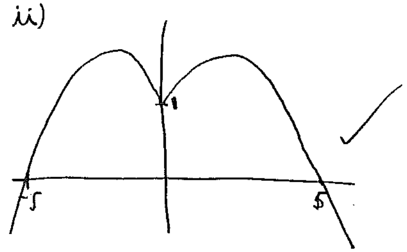
$\frac{dy}{dx} = \frac{-2xy^4 - 3x^2}{4x^2y^3}$

$= -\frac{2y^4+3x}{4xy^3}$

4 a) i)

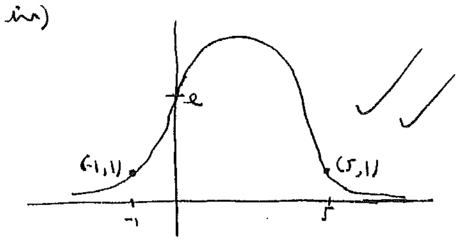



ii)

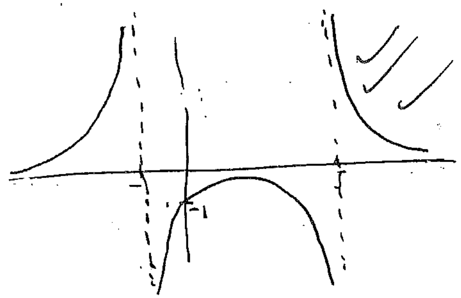



iii)


iv)




v)




b) $\frac{2\alpha^2}{\alpha\beta\gamma} = \frac{2\alpha}{\beta\gamma}$ $\alpha\beta\gamma = -\frac{-2}{1}$

$= 2$

$\therefore$ roots are $\alpha^2, \beta^2, \gamma^2$

$(\sqrt{x})^3 + 3(\sqrt{x})^2 - 2\sqrt{x} - 2 = 0$

$x\sqrt{x} + 3x - 2\sqrt{x} - 2 = 0$

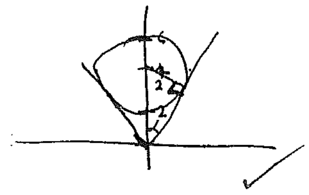
$x\sqrt{x} - 2\sqrt{x} = 2 - 3x$

$x(x-2)^2 = (2-3x)^2$

$x^3 - 4x^2 + 4x = 4 - 12x + 9x^2$

$x^3 - 13x^2 + 16x - 4 = 0$

c)




$\sin\alpha = \frac{2}{4}$

$\therefore \alpha = \frac{\pi}{6}$

greatest $\arg z = \frac{\pi}{2} + \frac{\pi}{6}$

$= \frac{2\pi}{3}$

least $\arg z = \frac{\pi}{2} - \frac{\pi}{6}$

$= \frac{\pi}{3}$

## Question Five

(a) If $1-i$, then $1+i$ must also be a root.

$\alpha+\beta+\gamma=-\frac{b}{a}$

$\alpha+1-i+1+i=-\frac{(-3)}{1}$

$\alpha=3-2=1$

Hence, roots are $1-i,1+i,1$.

(b)(i) $\alpha\beta=6$

$\alpha\beta\gamma\delta=48\Rightarrow\gamma\delta=8$

$(x^2+ax+6)(x^2+cx+8)=0$

coefficient of $x^3$ $\quad a+c=1$

coefficient of $x$ $\quad 8a+6c=-4$

$8a+8c=8$

$2a=-10\Rightarrow a=-5, c=6$

$(x^2-5x+6)(x^2+6x+8)=0$

(b)(ii) $(x-3)(x-2)(x+4)(x+2)=0$

$\therefore x=3,2,-4$ or $-2$

(c)(i) $P(x)=ax^4+4bx+c$

$P'(x)=4ax^3+4b$

double root at $x=\alpha$

$\Rightarrow P(\alpha)=P'(\alpha)=0$

$4a\alpha^3+4b=0$

$4a\alpha^3=-4b$

$\alpha^3=\frac{-4b}{4a}=-\frac{b}{a}$

(c)(ii) $P(\alpha)=0$

$a\alpha^4+4b\alpha+c=0$

$\alpha(a\alpha^3+4b)=-c$

$\alpha\left(a\times-\frac{b}{a}+4b\right)=-c$

$\alpha(-b+4b)=-c$

$\alpha^3(3b)^3=-c^3$

$-\frac{b}{a}\times27b^3=-c^3$

$\therefore ac^3=27b^4$

(c)(iii) $a=27, b=-8, c=16$

$\alpha^3=\frac{8}{27}\quad\therefore\alpha=\frac{2}{3}$

$27x^4-32x+16=(3x-2)^2(3x^2+4x+4)$

$x=\frac{-4\pm\sqrt{16-4(12)}}{6}=\frac{-4\pm\sqrt{-32}}{6}$

$\therefore x=\frac{2}{3},\frac{2}{3},\frac{-2\pm2\sqrt{2}i}{3}$

## Question Six

(a)(i) $m_{PQ}=\dfrac{\frac{1}{p}-\frac{1}{q}}{p-q}=\dfrac{q-p}{pq(p-q)}=-\dfrac{1}{pq}$

$y-\frac{1}{p}=-\frac{1}{pq}(x-p)$

$pqy-q=-x+p$

$x+pqy-(p+q)=0$

(a)(ii) height (distance from $O$ to $PQ$)

$h=\dfrac{|0+(pq)0-(p+q)|}{\sqrt{1^2+(pq)^2}}$

$=\dfrac{|p+q|}{\sqrt{1+p^2q^2}}$

$PQ=\sqrt{(p-q)^2+\left(\frac{1}{p}-\frac{1}{q}\right)^2}$

$=\sqrt{(p-q)^2+\frac{1}{(pq)^2}(q-p)^2}$

$=\sqrt{(p-q)^2\left(1+\frac{1}{p^2q^2}\right)}$ as $(q-p)^2=(p-q)^2$

$=\sqrt{\frac{(p-q)^2}{p^2q^2}}\times\sqrt{p^2q^2+1}$

$=\left|\frac{p-q}{pq}\right|\times\sqrt{1+p^2q^2}$

Area $=\frac{1}{2}\times\frac{|p+q|}{\sqrt{1+p^2q^2}}\times\left|\frac{p-q}{pq}\right|\times\sqrt{1+p^2q^2}$

$=\frac{|p^2-q^2|}{2|pq|}$ units$^2$

(b)(i) $e^2=1+\frac{9}{16}\quad\therefore e=\frac{5}{4}$

Focus $=(\pm ae,0)=(\pm5,0)$

(b)(ii) differentiate wrt $x$

$\frac{2x}{16}-\frac{2y}{9}\times\frac{dy}{dx}=0$

$\frac{dy}{dx}=\frac{x}{8}\times\frac{9}{2y}=\frac{9x}{16y}$

(b)(ii). continued at $P$ $\quad m_T=\frac{9x_1}{16y_1}$

equation of tangent :

$y-y_1=\frac{9x_1}{16y_1}(x-x_1)$

$16y_1(y-y_1)=9x_1(x-x_1)$

$16yy_1-16(y_1)^2=9xx_1-9(x_1)^2$

$\frac{xx_1}{16}-\frac{yy_1}{9}=\frac{(x_1)^2}{16}-\frac{(y_1)^2}{9}$

i.e. $\frac{xx_1}{16}-\frac{yy_1}{9}=1$ as $(x_1,y_1)$ lies on $\frac{x^2}{16}-\frac{x^2}{9}=1$

(b)(iii) $T$ is $x-$int. where $y=0$

$T=\left(\frac{16}{x_1},0\right)$

$\frac{S'T}{ST}=\dfrac{5+\frac{16}{x_1}}{5-\frac{16}{x_1}}$

i.e. $\frac{S'T}{ST}=\frac{5x_1+16}{5x_1-16}$

$\frac{S'P}{SP}=\frac{ePM'}{ePM}$ where $M$ is corr. directrix

$=\frac{PM'}{PM}=\dfrac{x_1+\frac{16}{5}}{x_1-\frac{16}{5}}$

i.e. $\frac{S'P}{SP}=\frac{5x_1+16}{5x_1-16}=\frac{S'T}{ST}$

(b)(iv) Let $\angle S'PT=\beta, \angle SPT=\gamma$ and $\angle PTS=\alpha$

$\angle PTS'=180-\alpha$ (st. $\angle$)

$\frac{\sin(180-\alpha)}{S'P}=\frac{\sin\beta}{S'T}$

i.e $\sin\beta=\frac{S'T\sin(180-\alpha)}{S'P}$

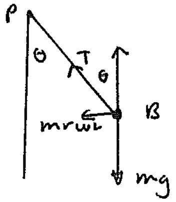
$=\frac{S'T\sin\alpha}{S'P}$

Similarly, $\sin\gamma=\frac{ST\sin\alpha}{SP}$

Since $\frac{S'T}{ST}=\frac{S'P}{SP}$ then $\frac{S'T}{S'P}=\frac{ST}{SP}$

Hence $\sin\beta=\sin\gamma$ i.e. $\beta=\gamma$

a)




i) $\omega = g$

Resolving Horizontally

$T\sin\theta = mr\omega^2$

$T\sin\theta = mrg^2$ # ①  (1)

ii) Balancing Vertically

$T\cos\theta = mg$ ②

① ÷ ② $\dfrac{T\sin\theta}{T\cos\theta} = \dfrac{mrg^2}{mg}$

$\tan\theta = gr$

$\therefore \theta = \tan^{-1}(gr)$ #  (1)

iii) From ① $T^2\sin^2\theta = m^2r^2g^4$ ③

from ② $T^2\cos^2\theta = m^2g^2$ ④

③ + ④ $T^2(\sin^2\theta + \cos^2\theta) = m^2g^2 + m^2r^2g^4$

$T^2 = m^2g^2(1 + g^2r^2)$

$T = mg\sqrt{1 + g^2r^2}$  (2)

b) i) $\cos 4\theta + i\sin 4\theta = (\cos\theta + i\sin\theta)^4$ [De Moivre Th]

$RHS = \cos^4\theta + 4i\cos^3\theta\sin\theta - 6\cos^2\theta\sin^2\theta - 4i\cos\theta\sin^3\theta + \sin^4\theta$

Equate Re $\cos 4\theta = \cos^4\theta - 6\cos^2\theta\sin^2\theta + \sin^4\theta$ ①

Equate Im $\sin 4\theta = 4\cos^3\theta\sin\theta - 4\cos\theta\sin^3\theta$ ②  (2)

ii) ② ÷ ① $\tan 4\theta = \dfrac{4\cos^3\theta\sin\theta - 4\cos\theta\sin^3\theta}{\cos^4\theta - 6\cos^2\theta\sin^2\theta + \sin^4\theta}$

(dividing top & bottom by $\cos^4\theta$)

$= \dfrac{4t - 4t^3}{1 - 6t^2 + t^4}$  (1)

7b) iii) $\tan 4\theta = 1$

$4\theta = \frac{\pi}{4}, \frac{5\pi}{4}, \frac{9\pi}{4}, \frac{13\pi}{4}$

$\theta = \frac{\pi}{16}, \frac{5\pi}{16}, \frac{9\pi}{16}, \frac{13\pi}{16}$

roots are $\tan\frac{\pi}{16}$, $\tan\frac{5\pi}{16}$

$\tan\frac{9\pi}{16} \Rightarrow -\tan\frac{7\pi}{16}$

$\tan\frac{13\pi}{16} \Rightarrow -\tan\frac{3\pi}{16}$  (3)

ie $\tan\frac{\pi}{16}, \tan\frac{5\pi}{16}, -\tan\frac{7\pi}{16}, -\tan\frac{3\pi}{16}$ or equivalent

iv) Product of roots $= 1$

$\therefore \tan\frac{\pi}{16} \times \tan\frac{5\pi}{16} \times (-\tan\frac{7\pi}{16}) \times (-\tan\frac{3\pi}{16}) = 1$

or $\tan\frac{\pi}{16}\tan\frac{3\pi}{16}\tan\frac{5\pi}{16}\tan\frac{7\pi}{16} = 1$  (2)

(Note Hence firm, A must follow from iii)

c) $I = \int_0^{\pi} x\cos 2x\,dx$ put $\pi - x$ in place of $x$

$I = \int_0^{\pi} (\pi - x)\cos(2\pi - 2x)\,dx$

$I = \int_0^{\pi} \pi\cos(2\pi - 2x)\,dx - \int_0^{\pi} x\cos(2\pi - 2x)\,dx$

$I = \int_0^{\pi} \pi\cos(2x)\,dx - \int_0^{\pi} x\cos(2x)\,dx$  ~~(2)~~ (3)

$= \pi\int_0^{\pi}\cos 2x\,dx - I$

$2I = -\pi[\sin 2x]_0^{\pi}$

$= -\pi[0 - 0]$

$= 0$

$\therefore I = 0$

let $u = x$ $v' = \cos 2x$

$u' = 1$ $v = \frac{1}{2}\sin 2x$

$I = \left[\frac{x}{2}\sin 2x\right]_0^{\pi} - \int_0^{\pi}\frac{1}{2}\sin 2x$

$= 0 - \left[-\frac{1}{4}\cos 2x\right]_0^{\pi}$

$= 0 + \frac{1}{4}[1 - 1]$

$= 0$

Question Eight

a) i) $I_n = \int_0^{\frac{\pi}{2}} (\sin x)^n \, dx$

$= \int_0^{\frac{\pi}{2}} (\sin x)(\sin)^{n-1} x \, dx$

let $u = \sin^{n-1} x$ $\quad \dot{v} = \sin x$

$\dot{u} = (n-1)(\sin x)^{n-2} \cos x$ $\quad v = -\cos x$ ✓

$I_n = -[\cos x \sin^{n-1} x]_0^{\frac{\pi}{2}} + (n-1)\int_0^{\frac{\pi}{2}} \sin^{n-2} x \cos^2 x \, dx$

$= 0 + (n-1)\int_0^{\frac{\pi}{2}} (\sin^{n-2} x)(1-\sin^2 x) \, dx$ ✓

$= (n-1)\left[\int_0^{\frac{\pi}{2}} \sin^{n-2} x \, dx - \int_0^{\frac{\pi}{2}} \sin^n x \, dx\right]$

$= (n-1)[I_{n-2} - I_n]$ ✓

$I_n = (n-1) I_{n-2} - (n-1) I_n$ ③

$n I_n = (n-1) I_{n-2}$

$I_n = \frac{n-1}{n} I_{n-2}$

ii) put $2n$ in place of $n$

$I_{2n} = \frac{2n-1}{2n} I_{2n-2}$ ✓

$= \frac{2n-1}{2n} \times \frac{2n-3}{2n-2} \times I_{2n-4} \dots$

$= \frac{2n-1}{2n} \times \frac{2n-3}{2n-2} \times \frac{2n-5}{2n-4} \times \dots I_2 \times I_0$ ③

and $I_0 = \int_0^{\frac{\pi}{2}} 1 \, dx$ $\quad , I_2 = \int_0^{\frac{\pi}{2}} \sin^2 x \, dx$

$= \frac{\pi}{2}$ ✓ $\quad = \frac{1}{2}[1-\cos 2x]_0^{\frac{\pi}{2}}$

$= \frac{1}{2}[(1--1)-(1-0)]$

$= \frac{1}{2}$

ie $I_{2n} = \frac{2n-1}{2n} \times \frac{2n-3}{2n-2} \times \frac{2n-5}{2n-4} \times \dots \times \frac{1}{2} \times \frac{\pi}{2}$

Now numerator & denominator increase by 2 from right to left

$\therefore I_{2n} = \frac{2n-1}{2n} \times \frac{2n-3}{2n-2} \times \frac{2n-5}{2n-4} \times \dots \frac{5}{6} \times \frac{3}{4} \times \frac{1}{2} \times \frac{\pi}{2}$

Q8 b) $\cos(A+B) + \cos(A-B) = 2\cos A \cos B$

$\cos(4x+x) + \cos(4x-x) = 2\cos 4x \cos x$

$\therefore$ eqn becomes

$2\cos 4x \cos x + 2\cos x = 0$ ✓

$2\cos x(\cos 4x + 1) = 0$

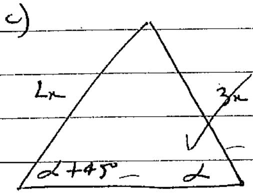
$2\cos x = 0$ $\quad , \cos 4x + 1 = 0$ $\quad 0 \le x \le \frac{\pi}{2}$, $0 \le 4x \le 2\pi$

$\cos x = 0$ $\quad \cos 4x = -1$

$x = \frac{\pi}{2}$ $\quad 4x = \pi$

✓ $\quad x = \frac{\pi}{4}$ ✓ ③

c)




$\frac{2x}{\sin \alpha} = \frac{3x}{\sin(\alpha + 45°)}$ ✓

$\frac{\sin(\alpha+45°)}{\sin \alpha} = \frac{3}{2}$

$3\sin \alpha = 2\sin(\alpha + 45°)$

$3\sin \alpha = 2(\sin \alpha \cos 45° + \cos \alpha \sin 45°)$

$3\sin \alpha = 2\left(\frac{\sin \alpha}{\sqrt{2}} + \frac{\cos \alpha}{\sqrt{2}}\right)$

$3\sin \alpha = \sqrt{2}\sin \alpha + \sqrt{2}\cos \alpha$

$\sin \alpha(3-\sqrt{2}) = \sqrt{2}\cos \alpha$ ✓

$\frac{\sin \alpha}{\cos \alpha} = \frac{\sqrt{2}}{3-\sqrt{2}} \times \frac{3+\sqrt{2}}{3+\sqrt{2}}$ ④

$\tan \alpha = \frac{3\sqrt{2}+2}{7}$

$\alpha = \tan^{-1}\left(\frac{2+3\sqrt{2}}{7}\right)$ ✓

1 mark for some working or diagram

8 d) (1), (2, 3), (4, 5, 6), (7, 8, 9, 10)

$T_1$ in first brackets = 1

$T_1$ in 2nd " = 1 + 1

$T_1$ in 3rd " = 1 + 1 + 2

$T_1$ in 4th " = 1 + 1 + 2 + 3

$T_1$ in 5th " = 1 + 1 + 2 + 3 + 4

ie $T_1$ in nth " $= 1 + \underbrace{(1+2+3+\cdots+n-1)}$ $S_n = \frac{n}{2}(a+l)$

ie $= 1 + \frac{n-1}{2}(1 + n - 1)$

$= 1 + \frac{n^2-n}{2}$

ie $a = \frac{n^2-n+2}{2}$ ✓ (2)

$S_n = \frac{n}{2}(2a + (n-1)d)$

Sum of n terms $= \frac{n}{2}\left(2\left(\frac{n^2-n+2}{2}\right) + (n-1)1\right)$ ✓

$= \frac{n}{2}(n^2 - n + 2 + n - 1)$

$= \frac{n}{2}(n^2+1)$