Sydney Girls High School

2003

TRIAL HIGHER SCHOOL CERTIFICATE EXAMINATION

Mathematics

Extension 2

This is a trial paper ONLY. It does not necessarily reflect the format or the contents of the 2003 HSC Examination Paper in this subject.

General Instructions

- Reading Time – 5 mins
- Working Time – 3 hours
- Attempt ALL questions
- ALL questions are of equal value
- All necessary working should be shown in every question. Marks may be deducted for careless or badly arranged work.
- Standard integrals are supplied
- Board-approved calculators may be used.
- Diagrams are not to scale
- Each question attempted should be started on a new sheet. Write on one side of the paper only.

Question 1.

a) Evaluate i) $\int_0^2 \frac{x}{x^2+4}.dx$

ii) $\int_{-\frac{\pi}{3}}^{\frac{\pi}{3}} \sin 2x.\cos x.dx$

iii) $\int_1^2 x^2 \log_e x.dx$

b) Let n be a positive integer, and let $I_n = \int_1^2 (\log_e x)^n.dx$.

prove that $I_n = 2(\log_e 2)^n - n.I_{n-1}$ and hence evaluate

$\int_1^2 (\log_e x)^3.dx$ as a polynomial in $\log_e 2$

Question 2.

a) i) Find $\sqrt{-3-4i}$

ii) Solve the equation $x^2 - 3x + 3 + i = 0$ over the complex field

b) i) Show that there are two complex numbers z such that

$|z - 2 - i| = 1$ and $\arg z = \frac{\pi}{4}$,

ii) Find the moduli of the two values of z found in part i)

c) A point P representing the complex number z moves in the Argand Diagram so this it lies in the region defined by:

$|z - 1| \leq |z - i|$ and $|z - 2 - 2i| \leq 1$

i) Indicate on a sketch, the region within which P lies

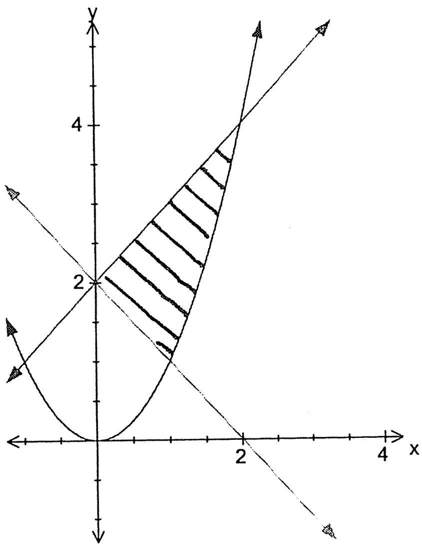
ii) If P describes the boundary of the region, find

α) the value of $|z|$ when arg z has its smallest value

β) the values of z in the form a + ib when $\arg(z-1) = \frac{\pi}{4}$

Question 3:

a) The adjacent diagam shows the area enclosed by $y = 2-x$, $y = 2+ x$ and $y = x^2$. The area is to be rotated about the Y axis.



i) Find the shaded area

ii) Find the volume that is formed when it is rotated

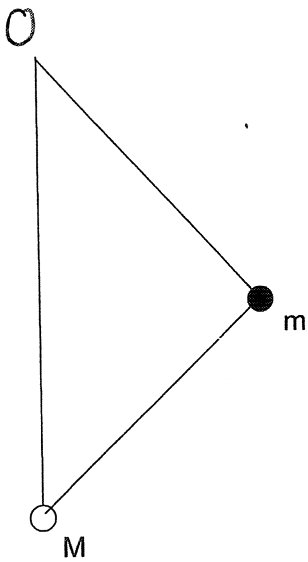
b) A satellite moves in a circular orbit of radius 8000km, making 12 revolutions per day. Find:

i) the velocity of the satellite

ii) the centripetal force acting on the satellite if the mass of that satellite is 500kg.

c) A particle of mass m is attached to a fixed point O by a string of length one metre, and by another string of the same length to a small ring of mass M which can slide on a smooth vertical wire underneath O.

If m describes a horizontal circle with constant angular velocity w, prove that its depth below O is $\left(\frac{m+2M}{mw^2}\right)g$, where g is acceleration due to gravity



Questions 4:

a) (i) Sketch $\frac{x^2}{4}+\frac{y^2}{9}=2$, indicating the centre, foci and directrices

(ii) If $P(2\sqrt{2}\cos\theta, 3\sqrt{3}\sin\theta)$ lies on the ellipse find

α) The equation of the normal at P

β) The value of θ to the nearest degree if the normal passes through the point $(-2\sqrt{2},0)$

b) $P\left(3p, \frac{3}{p}\right)$ and $Q\left(3q, \frac{3}{q}\right)$ lie on the hyperbola $xy = 9$.

(i) Find the equation of the tangent at P

(ii) Find the point of intersection T, of the tangents at P and Q.

(iii) If the chord of contact from T passes through the point (0,2) find the locus of T.

Question 5.

a) Given $f(x) = \dfrac{7x}{(x^2+3)(x+2)}$

i) Express f(x) as a sum of partial fractions

ii) Evaluate $\int_0^3 f(x).dx$

b) Without the use of calculus, sketch the following curves

i) $y = \dfrac{x(x-2)}{x-1}$ ii) $y = \dfrac{x(x-1)}{x-2}$

c) Consider $y = \dfrac{x^3}{(x-1)^2}$

i) Determine the asymptotes

ii) Determine the stationary points

iii) Sketch the curve showing any important features

Question 6.

a) (i) Show that if a polynomial P(x) has a root b of multiplicity m, the the polynomial P'(x) has the root b with multiplicity m-1

(ii) Given that $Q(x) = x^4 - 5x^3 + 4x^2 + 3x + 9$ has a zero of multiplicity 2, solve the equation $Q(x) = 0$ over the complex field

b) If α, β and γ are the roots of $3x^3 + 4x^2 + 5x + 1 = 0$, find the value of $\dfrac{1}{\alpha^2\beta^2} + \dfrac{1}{\beta^2\gamma^2} + \dfrac{1}{\gamma^2\alpha^2}$

c) i) If $x^2 - 3x + 4 = 0$, show that $x^4 = 3x - 20$

ii) Hence or otherwise, find the equation with roots α^4 and β^4 if the roots of $x^2 - 3x + 4 = 0$ are α and β.

Question 7:

a) Show that the volume of the largest cylinder that can be cut from a solid sphere of radius r cm is $\dfrac{4\pi r^3}{3\sqrt{3}}$ cm^3

b) (i) Find the five roots of $z^5 = 1$ and write them in mod-arg form.

(ii) Show that when these five roots are plotted on an Argand Diagram, they form the vertices of a regular pentagon of area $\dfrac{5}{2}\sin\dfrac{2\pi}{5}$

(iii) Factorise $z^5 - 1$ over the real field

(iv) Deduce that $\cos\dfrac{2\pi}{5}$ is a root of the equation $4x^2 + 2x - 1 = 0$ and hence find the exact value of $\cos\dfrac{2\pi}{5}$.

Question 8:

a) A particle of mass m falls from rest at a height h above the earth's surface, against a resistance kv per unit mass when its speed is v; k being a positive constant.

(i) Show that its equation of motion may be written in the form

$$v\frac{dv}{dx} = g - kv$$

(ii) If the particle reaches the surface of the earth with speed V, show that

$$\log_e\left(1-\frac{kV}{g}\right) + \frac{kV}{g} + \frac{k^2h}{g} = 0$$

b) (i) A particle P is projected from a point O on horizontal ground, with speed V at an angle $\theta = \tan^{-1}\left(\dfrac{1}{3}\right)$. The particle passes through the point with co-ordinates $\left(3a, \dfrac{3a}{4}\right)$. Show that $V^2 = 20ga$.

(ii) A particle Q is projected from the same point O at the instant when P reaches its maximum height. It strikes the ground at the same place and time as P strikes the ground. Show that the speed of projection of Q is $\sqrt{\dfrac{145ga}{2}}$ and find the tangent of the angle of projection.