SYDNEY GRAMMAR SCHOOL

2013 Trial Examination

FORM VI

MATHEMATICS EXTENSION 2

Thursday 1st August 2013

General Instructions

- Reading time — 5 minutes
- Writing time — 3 hours
- Write using black or blue pen.
- Board-approved calculators and templates may be used.
- A list of standard integrals is provided at the end of the examination paper.

Total — 100 Marks

- All questions may be attempted.

Section I – 10 Marks

- Questions 1–10 are of equal value.
- Record your solutions to the multiple choice on the sheet provided.

Section II – 90 Marks

- Questions 11–16 are of equal value.
- All necessary working should be shown.
- Start each question in a new booklet.

Collection

- Write your candidate number on each booklet and on your multiple choice answer sheet.
- Hand in the booklets in a single well-ordered pile.
- Hand in a booklet for each question in Section II, even if it has not been attempted.
- If you use a second booklet for a question, place it inside the first.
- Place your multiple choice answer sheet inside the answer booklet for Question Eleven.
- Write your candidate number on this question paper and submit it with your answers.

Checklist

- SGS booklets — 6 per boy
- Multiple choice answer sheet
- Candidature — 69 boys

Examiner
DS/REP

SECTION I - Multiple Choice

Answers for this section should be recorded on the separate answer sheet handed out with this examination paper.

QUESTION ONE

Which function is a primitive of $\dfrac{2x}{2x-1}$?

(A) $x + \ln(2x-1)$

(B) $\ln(2x-1)$

(C) $x + \frac{1}{2}\ln(2x-1)$

(D) $\frac{1}{2}\ln(2x-1)$

QUESTION TWO

Which expression is a correct factorisation of $z^3 - i$?

(A) $(z-i)(z^2+iz+1)$

(B) $(z+i)(z^2-iz-1)$

(C) $(z+i)(z-i)^2$

(D) $(z+i)^3$

QUESTION THREE

If $f(x)$ is an odd function, which statement

(A) $\displaystyle\int_{-a}^{a} f(x)\,dx = 2\int_{0}^{a} f(x)\,dx$

(B) $\displaystyle\int_{0}^{2a} f(x)\,dx = 2\int_{0}^{a} f(x)\,dx$

(C) $\displaystyle\int_{-2a}^{a} f(x)\,dx = \int_{-a}^{2a} f(x)\,dx$

(D) $\displaystyle\int_{-a}^{2a} f(x)\,dx = \int_{a}^{2a} f(x)\,dx$

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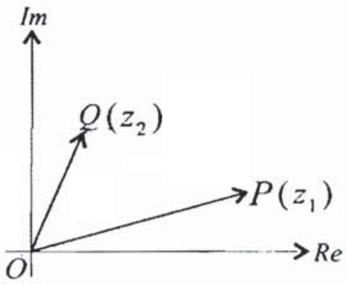
QUESTION FOUR



The points P and Q in the first quadrant represent the complex numbers z_1 and z_2 respectively, as shown in the diagram above. Which statement about the complex number $z_2 - z_1$ is true?

(A) It is represented by the vector QP.

(B) Its principal argument lies between $\frac{\pi}{2}$ and π.

(C) Its real part is positive.

(D) Its modulus is greater than $|z_1 + z_2|$.

QUESTION FIVE

An ellipse has foci at $(-6, 0)$ and $(6, 0)$, and its directrices have equations $x = -8$ and $x = 8$. What is the eccentricity of the ellipse?

(A) $\frac{1}{2}\sqrt{3}$ (B) $\frac{1}{3}\sqrt{3}$ (C) $\frac{2}{3}\sqrt{3}$ (D) $3\sqrt{3}$

QUESTION SIX

The polynomial $P(x) = x^3 + 3x^2 - 24x + 28$ has a double zero. What is the value of the double zero?

(A) -7 (B) -4 (C) 4 (D) 2

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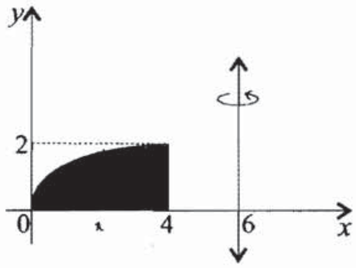
QUESTION SEVEN



The diagram above shows the region bounded by the curve $y = \sqrt{x}$ and the x-axis, from $x = 0$ to $x = 4$. The region is rotated about the line $x = 6$ to form a solid of revolution. Which integral gives the volume of the solid?

(A) $\displaystyle\int_0^2 \pi(4 - y^2)(8 - y^2)\,dy$

(B) $\displaystyle\int_0^2 4\pi(5 - y^2)\,dy$

(C) $\displaystyle\int_0^2 \pi(4 - y^2)(6 - y^2)\,dy$

(D) $\displaystyle\int_0^2 \pi(2 - y^2)(6 - y^2)\,dy$

QUESTION EIGHT

The curve defined by the equation $x^2 - xy + 2y^2 = 4$ passes through the point $P(1, -1)$. What is the gradient of the tangent to the curve at P?

(A) $\frac{1}{2}$ (B) $-\frac{1}{5}$ (C) $\frac{3}{5}$ (D) $\frac{3}{4}$

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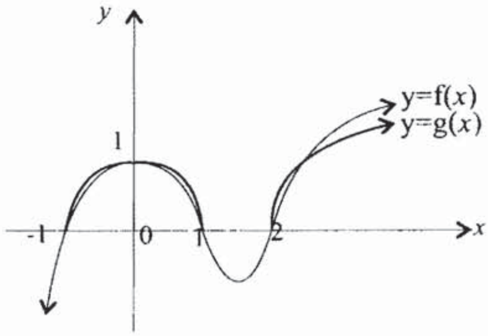
QUESTION NINE



In the diagram above, the relationship between the functions $f(x)$ and $g(x)$ could be represented by:

(A) $g(x) = \left(f(x)\right)^2$

(B) $g(x) = \log_e f(x)$

(C) $g(x) = \sqrt{f(x)}$

(D) $g(x) = |f(x)|$

QUESTION TEN

Without attempting to evaluate the integrals, determine which of the following inequalities is FALSE:

(A) $\displaystyle\int_1^2 \frac{1}{1+x}\,dx < \int_1^2 \frac{1}{x}\,dx$

(B) $\displaystyle\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{\sin x}{x}\,dx < \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{1}{x}\,dx$

(C) $\displaystyle\int_0^{\frac{\pi}{4}} \tan^2 x\,dx < \int_0^{\frac{\pi}{4}} \tan^3 x\,dx$

(D) $\displaystyle\int_1^2 e^{-x^2}\,dx < \int_0^1 e^{-x^2}\,dx$

End of Section I

Exam continues overleaf ...

SECTION II - Written Response

Answers for this section should be recorded in the booklets provided.

Show all necessary working.

Start a new booklet for each question.

QUESTION ELEVEN (15 marks) Use a separate writing booklet. **Marks**

(a) Solve the quadratic equation $4z^2 + 4z + 5 = 0$. **2**

(b) Find the real values of x and y for which $\dfrac{x}{i} - \dfrac{y}{1+i} = -1 - 3i$. **2**

(c) Find $\displaystyle\int \frac{3x}{2x^2 - 5x + 2}\, dx$. **3**

(d) Find $\displaystyle\int \frac{1}{\sqrt{x^2 + 6x + 34}}\, dx$. **2**

(e) Evaluate $\displaystyle\int_0^{\frac{\pi}{3}} \tan^4 x \, dx$. **3**

(f) (i) Use the substitution $u = a - x$ to prove that $\displaystyle\int_0^a f(x)\, dx = \int_0^a f(a-x)\, dx$. **1**

(ii) Hence find the value of $\displaystyle\int_0^1 x(1-x)^7 \, dx$. **2**

Exam continues next page ...

QUESTION TWELVE (15 marks) Use a separate writing booklet. **Marks**

(a) Shade the region in the Argand diagram where $|z + 3i| > 2|z|$. **3**

(b) (i) Express $-2 + 2i$ in modulus-argument form. **1**

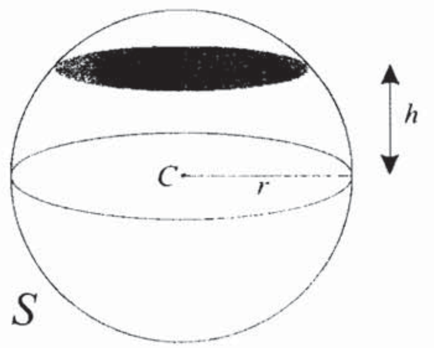
(ii) Simplify $(-2 + 2i)^{8k}$, where k is an integer. **2**

(c) A complex number z satisfies $\arg(z - 1) = \frac{\pi}{6}$.

(i) Sketch the locus of z. **1**

(ii) Show that $|z - 5| \geq 2$. **1**

(d)



In the diagram above, S is a sphere of radius r. The point C is the centre of the sphere. A typical horizontal cross-section h units above C is shown.

(i) Find the area of this cross-section as a function of h. **1**

(ii) Hence prove that the volume of S is $\frac{4}{3}\pi r^3$. **2**

(e) The polynomial $P(x) = x^4 - 4x^3 + 10x^2 - 12x - 40$ has zeroes α, β, γ and δ.

(i) Find a polynomial with zeroes $\alpha - 1$, $\beta - 1$, $\gamma - 1$ and $\delta - 1$. **2**

(ii) Hence find the zeroes of $P(x)$. **2**

Exam continues overleaf ...

QUESTION THIRTEEN (15 marks) Use a separate writing booklet. **Marks**

(a) (i) Write down the equation of a line with gradient m that passes through the point $T(-4c, 2c)$. **1**

(ii) Solve the equation in part (i) simultaneously with the equation $xy = c^2$ to obtain a quadratic equation in x. **2**

(iii) Hence find the gradients of the two tangents to the rectangular hyperbola $xy = c^2$ that pass through the point $T(-4c, 2c)$. **2**

(b) Consider the polynomial $P(z) = z^4 + (1 - 2i)z^2 - 2i$.

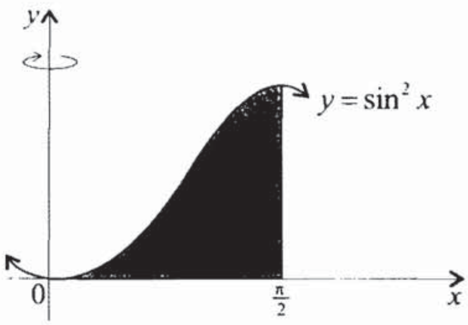
(i) Show that $P(i) = 0$. **1**

(ii) Explain why $P(-i)$ must also be zero. **1**

(iii) Suppose that the other two zeroes of $P(z)$ are w and $-w$. Use the product of the zeroes to find w. **3**

(c) (i) Find $\int x \cos 2x \, dx$. **2**

(ii) **3**



The diagram above shows the region bounded by the curve $y = \sin^2 x$, the x-axis and the line $x = \frac{\pi}{2}$. The region is rotated about the y-axis through 360°. Use the method of cylindrical shells to find the exact volume of the solid formed.

Exam continues next page ...

QUESTION FOURTEEN (15 marks) Use a separate writing booklet. **Marks**

(a) An object of mass 2 kg is projected vertically upwards at 20 m/s and experiences air resistance of magnitude $\frac{1}{10}v^2$ Newtons, where v is the speed of the object after t seconds. Take $g = 10\,\text{m/s}^2$.

(i) Show that the maximum height reached by the object is $10\ln 3$ metres. **3**

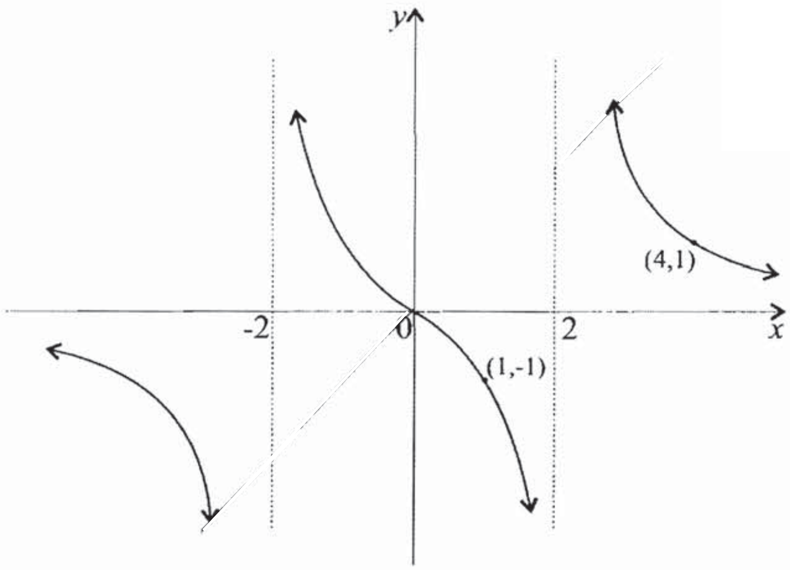
(ii) Find the speed of the object, correct to three significant figures, at the instant it reaches half its maximum height. **2**

(b) Let $I_n = \int_1^e (1+\ln x)^n \, dx$, where $n \geq 0$.

(i) Use integration by parts to show that $I_n = (2^n)\,e - 1 - n\,I_{n-1}$. **2**

(ii) Hence find the exact value of $\int_1^e (2+\ln x)(1+\ln x)^2 \, dx$. **2**

(c)



The diagram above shows the graph of the odd function $y = f(x)$, where

$$f(x) = \frac{3x}{x^2 - 4}.$$

Sketch the graphs of each of the following functions on large separate diagrams, showing the x-intercepts and asymptotes. You are NOT expected to find any stationary points.

(i) $y = \dfrac{1}{f(x)}$ **2**

(ii) $y = \log_e f(x)$ **2**

(iii) $y = x + f(x)$ **2**

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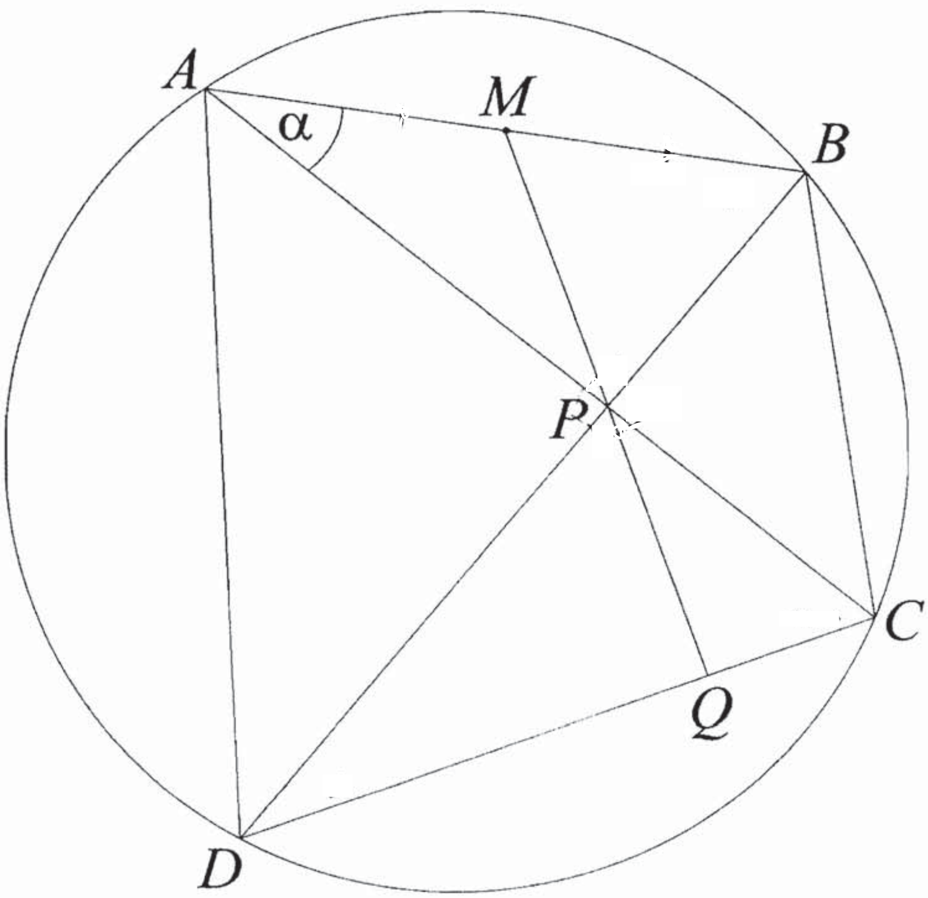
QUESTION FIFTEEN (15 marks) Use a separate writing booklet. **Marks**

(a)



In the diagram above $ABCD$ is a cyclic quadrilateral whose diagonals are perpendicular and intersect at P. Let M be the midpoint of AB, and suppose that MP produced meets DC at Q. Let $\angle PAM = \alpha$.

(i) Explain why $AM = PM$. **1**

(ii) Prove that $MQ \perp DC$. **3**

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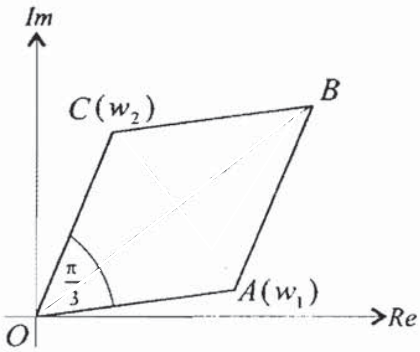
QUESTION FIFTEEN (Continued)

(b)



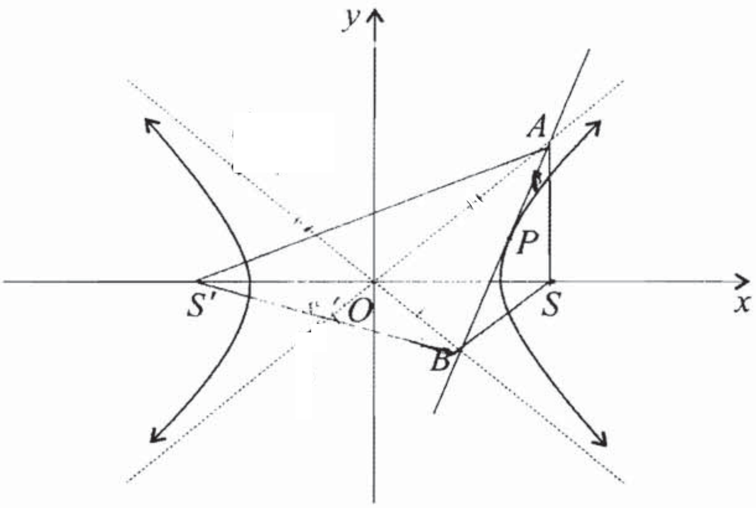
In the Argand diagram above, $OABC$ is a rhombus with $\angle COA = \frac{\pi}{3}$. The points A and C represent the complex numbers w_1 and w_2 respectively.

(i) Explain why $w_2 = w_1 \operatorname{cis} \frac{\pi}{3}$. **1**

(ii) Write down, in terms of w_1 only, the complex numbers represented by the vectors OB and AC. **2**

(iii) By considering $i(w_1 + w_2)$, show that the diagonals OB and AC of the rhombus are perpendicular. **2**

(c)



The diagram above shows the tangent at a point $P(a\sec\theta, b\tan\theta)$ on the hyperbola $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$ meeting the asymptotes of the hyperbola at A and B. The points S and S' are the foci of the hyperbola.

(i) Show that the tangent at P has equation $bx\sec\theta - ay\tan\theta = ab$. **1**

(ii) Show that $OA \times OB = a^2 + b^2$, where O is the origin. **2**

(iii) Extend AO to A' so that $OA' = OB$ and extend BO to B' so that $OB' = OA$. Explain why the points A, B, A' and B' are concyclic. **1**

(iv) Hence show that the points A, S, B and S' are concyclic. **2**

Exam continues overleaf ...

QUESTION SIXTEEN (15 marks) Use a separate writing booklet. **Marks**

(a) Let $z = \cos\theta + i\sin\theta$ and suppose that n is a positive integer.

(i) Show that $z^n + z^{-n} = 2\cos n\theta$. **1**

(ii) Hence use the identity $2\cos A\sin B = \sin(A+B) - \sin(A-B)$ to show that **2**

$$\left(z^{2n} + z^{2n-2} + z^{2n-4} + \cdots + z^{-2n}\right)\sin\theta = \sin(2n+1)\theta.$$

(iii) Use part (ii) and the identity $\cos 3A = 4\cos^3 A - 3\cos A$ to deduce that **1**

$$8\cos^3 2\theta + 4\cos^2 2\theta - 4\cos 2\theta - 1 = \frac{\sin 7\theta}{\sin\theta}.$$

(iv) Hence show that $\cos\frac{2\pi}{7}$ is a root of the equation $8x^3 + 4x^2 - 4x - 1 = 0$. **1**

(b) (i) Use a suitable double angle formula to show that $\tan\frac{\pi}{8} = \sqrt{2} - 1$. **1**

(ii) Find $\cos 4\theta$ in terms of $\cos\theta$. **1**

(iii) Let $I = \displaystyle\int_{-1}^{1} \frac{1}{\sqrt{1+x} + \sqrt{1-x} + 2}\,dx$.

(α) Show, by using the substitution $x = \sin 4\theta$, that **2**

$$I = \int_0^{\frac{\pi}{8}} \frac{2\cos 4\theta}{\cos^2\theta}\,d\theta.$$

(β) Hence find the exact value of I. **2**

(c) (i) Explain why **1**

$$\left(1 - \frac{1}{2}\right) + \left(\frac{1}{3} - \frac{1}{4}\right) + \left(\frac{1}{5} - \frac{1}{6}\right) + \cdots + \left(\frac{1}{2k-1} - \frac{1}{2k}\right) > \frac{k}{(2k+1)(2k+2)}.$$

(ii) Prove by mathematical induction, or otherwise, that for all $n \geq 2$, **3**

$$n\left(1 + \frac{1}{3} + \frac{1}{5} + \cdots + \frac{1}{2n-1}\right) > (n+1)\left(\frac{1}{2} + \frac{1}{4} + \frac{1}{6} + \cdots + \frac{1}{2n}\right).$$

——— End of Section II ———

END OF EXAMINATION