

QUESTION EIGHT (Start a new answer booklet)

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3 (a) Solve $|x^2 - 2x - 3| < 3x - 3$.

7 (b) Consider the integral $I_n = \int_0^1 x^{2n+1} e^{-x^2} dx$.

(i) Use integration by parts to show that $I_n = -\frac{1}{2e} + nI_{n-1}$, for $n \geq 1$.

(ii) Show that $I_0 = \frac{1}{2} - \frac{1}{2e}$.

(iii) Prove by mathematical induction that for all $n \geq 1$:

$$1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \cdots + \frac{1}{n!} = e - \frac{2e I_n}{n!}.$$

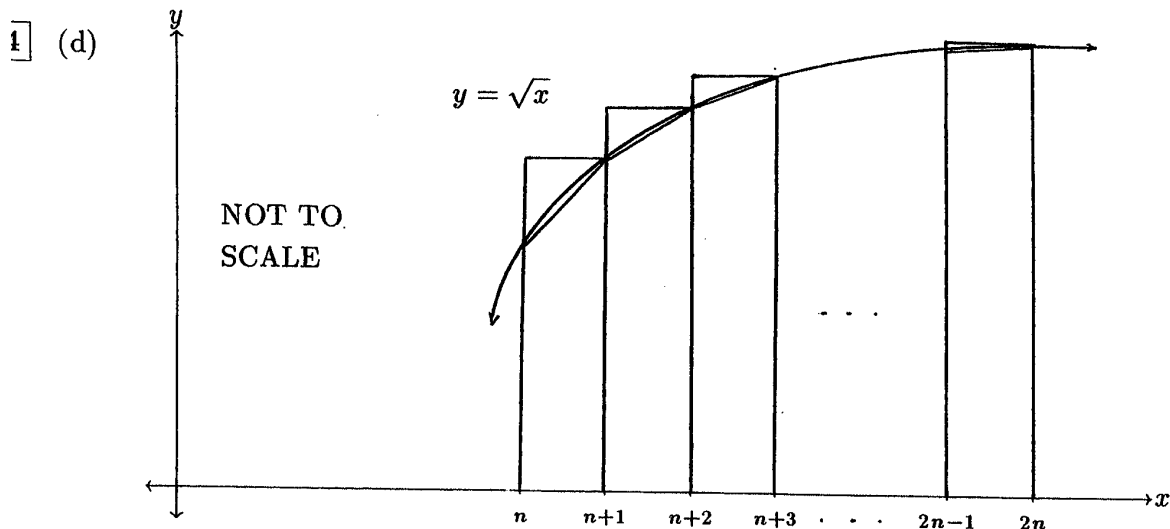
(iv) It is clear that $0 \leq I_n \leq 1$, because $0 \leq x^{2n+1} e^{-x^2} \leq 1$, for $0 \leq x \leq 1$.
Use this fact to deduce from the previous part that:

$$1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \cdots = e.$$

CARE: QUESTION EIGHT CONTINUES OVERLEAF

QUESTION EIGHT (Continued)

□ (c) Show that $\int_n^{2n} \sqrt{x} dx = \frac{2}{3}n\sqrt{n} (2\sqrt{2} - 1)$, where n is a positive integer.



The diagram above shows part of the graph of $y = \sqrt{x}$, drawn quite distorted so that the chords can be seen more clearly. For some positive integer n , ordinates have been drawn at $x = n, x = n + 1, \dots, x = 2n$. Upper rectangles and trapezia have been constructed as shown.

(i) Using the upper rectangles and part (c), show that:

$$\sqrt{n} + \sqrt{n+1} + \dots + \sqrt{2n} > \frac{2}{3}n\sqrt{n} (2\sqrt{2} - 1) + \sqrt{n}$$

(you may assume that $y = \sqrt{x}$ is an increasing function).

(ii) Using the trapezia and part (c), show that:

$$\sqrt{n} + \sqrt{n+1} + \dots + \sqrt{2n} < \frac{2}{3}n\sqrt{n} (2\sqrt{2} - 1) + \frac{1}{2} (\sqrt{n} + \sqrt{2n})$$

(you may assume that $y = \sqrt{x}$ is concave down).

(iii) Suppose it is claimed that the average of the numbers:

$$\sqrt{1\,000\,000}, \sqrt{1\,000\,001}, \dots, \sqrt{2\,000\,000}$$

is about 1218.9512. If one relies on the bounds established in parts (i) and (ii), what is the maximum possible error in this claim?

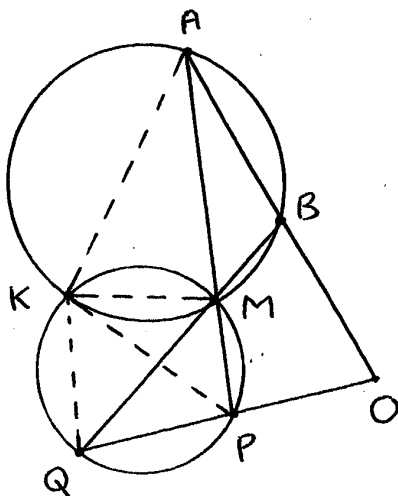
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Candidate Number:

Tear out this sheet of paper, write your candidate number at the top, and place the sheet inside your answer booklet for Question Seven.

QUESTION SEVEN

(b) This is a reprint of the diagram for part (b) of Question Seven.



(c) This is a reprint of the diagram for part (c) of Question Seven.

