



















**QUESTION EIGHT** (Start a new answer booklet)

arks

**3** (a) Solve  $|x^2 - 2x - 3| < 3x - 3$ .

**7** (b) Consider the integral  $I_n = \int_0^1 x^{2n+1} e^{-x^2} dx$ .

(i) Use integration by parts to show that  $I_n = -\frac{1}{2e} + nI_{n-1}$ , for  $n \geq 1$ .

(ii) Show that  $I_0 = \frac{1}{2} - \frac{1}{2e}$ .

(iii) Prove by mathematical induction that for all  $n \geq 1$ :

$$1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \cdots + \frac{1}{n!} = e - \frac{2e I_n}{n!}.$$

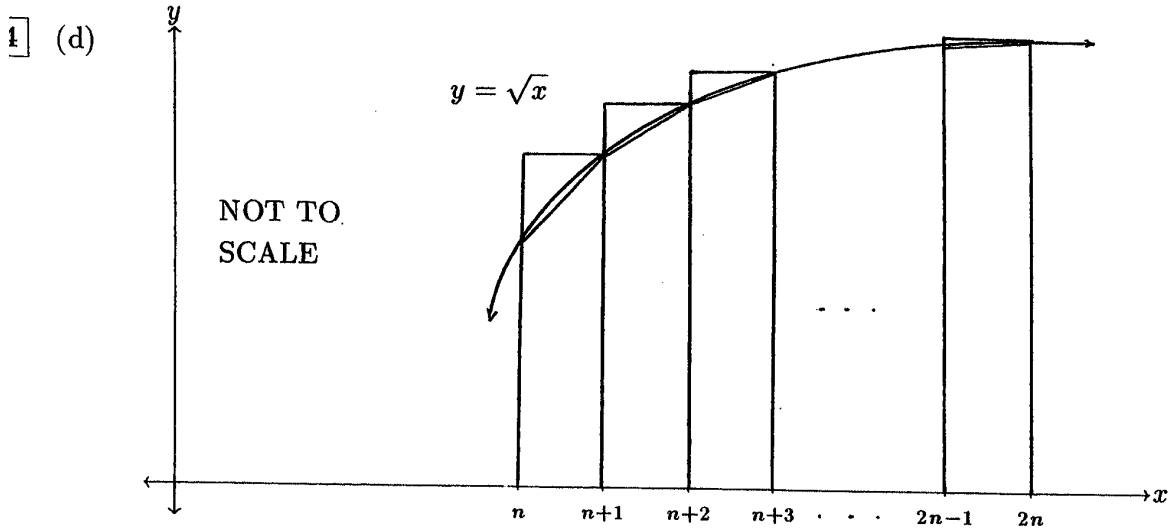
(iv) It is clear that  $0 \leq I_n \leq 1$ , because  $0 \leq x^{2n+1} e^{-x^2} \leq 1$ , for  $0 \leq x \leq 1$ .  
Use this fact to deduce from the previous part that:

$$1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \cdots = e.$$

CARE: QUESTION EIGHT CONTINUES OVERLEAF

**QUESTION EIGHT** (Continued)

□ (c) Show that  $\int_n^{2n} \sqrt{x} dx = \frac{2}{3}n\sqrt{n} (2\sqrt{2} - 1)$ , where  $n$  is a positive integer.



The diagram above shows part of the graph of  $y = \sqrt{x}$ , drawn quite distorted so that the chords can be seen more clearly. For some positive integer  $n$ , ordinates have been drawn at  $x = n, x = n + 1, \dots, x = 2n$ . Upper rectangles and trapezia have been constructed as shown.

(i) Using the upper rectangles and part (c), show that:

$$\sqrt{n} + \sqrt{n+1} + \dots + \sqrt{2n} > \frac{2}{3}n\sqrt{n} (2\sqrt{2} - 1) + \sqrt{n}$$

(you may assume that  $y = \sqrt{x}$  is an increasing function).

(ii) Using the trapezia and part (c), show that:

$$\sqrt{n} + \sqrt{n+1} + \dots + \sqrt{2n} < \frac{2}{3}n\sqrt{n} (2\sqrt{2} - 1) + \frac{1}{2} (\sqrt{n} + \sqrt{2n})$$

(you may assume that  $y = \sqrt{x}$  is concave down).

(iii) Suppose it is claimed that the average of the numbers:

$$\sqrt{1\,000\,000}, \sqrt{1\,000\,001}, \dots, \sqrt{2\,000\,000}$$

is about 1218.9512. If one relies on the bounds established in parts (i) and (ii), what is the maximum possible error in this claim?

WMP

