Student Number: . . . .

Set: 12ME2-1 (FES)

SHORE

2008

Trial HSC Examination

Mathematics Extension 2

General Instructions

- Reading time − 5 minutes
- Working time − 3 hours
- Write using black or blue pen
- Board-approved calculators may be used
- A table of standard integrals is provided on the back page of this question paper
- All necessary working should be shown in every question

Total marks − 120

- Attempt Questions 1 − 8
- All questions are of equal value
- Start each question in a new writing booklet
- Write your examination number on the front cover of each booklet to be handed in
- If you do not attempt a question, submit a blank booklet marked with your examination number and "N/A" on the front cover

DO NOT REMOVE THIS PAPER FROM THE EXAMINATION ROOM

Blank Page

Total Marks – 120
Attempt Questions 1 – 8
All Questions are of equal value

Begin each question on a NEW BOOKLET, writing your name and question number at the top of the page. Extra booklets are available.

Question 1 (15 marks) Use a SEPARATE writing booklet **Marks**

(a) Find $\int (e^{x}+e^{-x})^{2}\,dx$ **2**

(b) Find $\int \frac{2x^{2}-2x+1}{(x-2)(x^{2}+1)}\,dx$ **3**

(c) Use the substitution $t=\tan\frac{x}{2}$ to evaluate $\int_{0}^{\frac{\pi}{2}} \frac{1}{2+\sin x}\,dx.$ **5**

(d) Let $I_n=\int_0^1 \frac{1}{(1+x^2)^n}\,dx\ ;\ n=1,2,3,\ldots$

(i) Show that $I_{n+1}=\frac{2n-1}{2n}I_n+\frac{1}{n.2^{n+1}}\ ;\ n=1,2,3,\ldots$ **3**

(ii) Hence evaluate $\int_0^1 \frac{1}{(1+x^2)^3}\,dx.$ **2**

P.T.O……………………….

Question 2 **Use a SEPARATE writing booklet** **Marks**

(a) $z_1 = 1 + i$ and $z_2 = \sqrt{3} - i$.

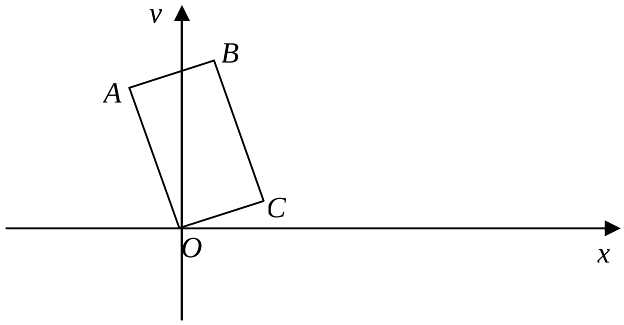
(i) Find $\frac{z_1}{z_2}$ in the form $a+ib$ where a and b are real. **1**

(ii) Write z_1 and z_2 in modulus – argument form. **2**

(iii) By equating equivalent expressions for $\frac{z_1}{z_2}$, write $\cos\frac{5\pi}{12}$ as a surd. **1**

(iv) Explain why there is no positive integer n such $z_1 z_2^n$ is real. **2**

(b)



The points $OABC$ are the vertices of a rectangle on the Argand diagram with $|OA| = 2|OC|$. If OC represents the complex number $p+iq$, write down the complex numbers represented by:

(i) $\overrightarrow{OA}$ **1**

(ii) $\overrightarrow{OB}$ **1**

(iii) $\overrightarrow{BC}$ **1**

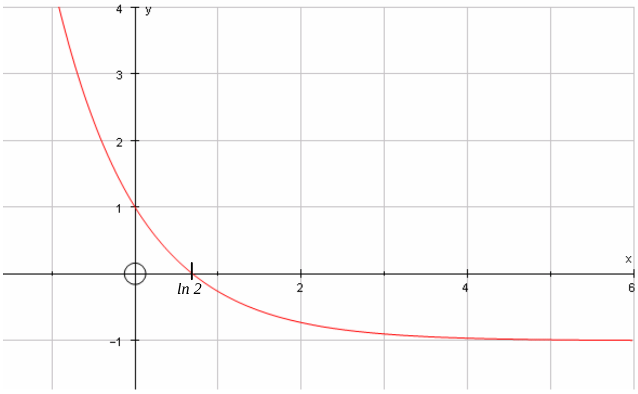
(iv) $\overrightarrow{AC}$ **1**

(c) (i) If $z = \cos\theta + i\sin\theta$, explain why $z^n + z^{-n} = 2\cos n\theta$ and $z^n - z^{-n} = 2i\sin n\theta$ for positive integers n. **2**

(ii) By considering the Binomial expansions of $(z + z^{-1})^3$ and $(z - z^{-1})^3$, show that $4(\cos^3\theta + \sin^3\theta) = (\cos 3\theta - \sin 3\theta) + 3(\cos\theta + \sin\theta)$. **3**

Question 3 **Use a SEPARATE writing booklet** **Marks**

(a)



The diagram shows the graph of $f(x) = 2e^{-x} - 1$. On separate diagrams sketch the following graphs, showing the intercepts on the axes and the equations of any asymptotes:

(i) $y = |f(x)|$. **1**

(ii) $y = \{f(x)\}^2$. **1**

(iii) $y = \dfrac{1}{f(x)}$. **2**

(iv) $y = ln\ \{f(x)\}$. **1**

(b) Consider the curve $y^2 = x^4(4 + x)$

(i) Sketch the curve. **2**

(ii) Find the area of the loop of the curve from $x = -4$ to $x = 0$. **3**

(c) The roots of $x^3 - 3x^2 - 2x + 4 = 0$ are α, β and γ. Answer the following without finding the actual values of α, β and γ.

(i) Find a cubic equation whose roots are α^2, β^2 and γ^2. **2**

(ii) Hence or otherwise find the value of $\alpha^2 + \beta^2 + \gamma^2$. **1**

(iii) Determine the value of $\alpha^3 + \beta^3 + \gamma^3$. **2**

P.T.O……………………..

Question 4 **Use a SEPARATE writing booklet** **Marks**

(a) Given $|z+i| \le 2$ and $0 \le arg(z+1) \le \frac{\pi}{4}$. Sketch the region in an Argand diagram which contains the point P representing z. **3**

(b) Consider the five 5^{th} roots of unity.

(i) Solve $z^5 - 1 = 0$ over the complex field giving your answers in modulus-argument form. **3**

(ii) Hence express $z^5 - 1$ as the product of real linear and quadratic factors. **3**

(iii) Write down the complex roots of $z^4 + z^3 + z^2 + z + 1 = 0$ giving your answers in modulus-argument form. **1**

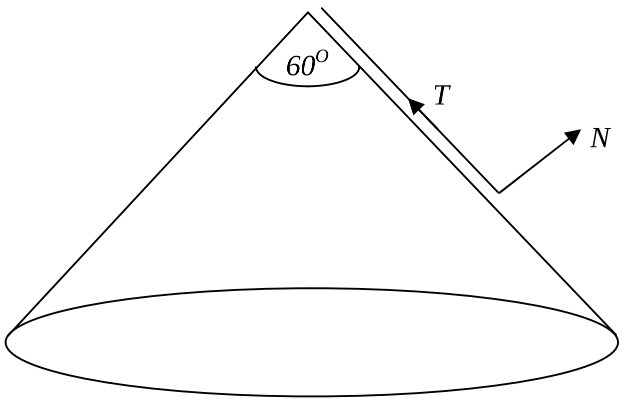
(iv) Hence prove that $\cos\frac{2\pi}{5} + \cos\frac{4\pi}{5} = -\frac{1}{2}$. **2**

(c) If $x > 0$ and $y > 0$ prove that $\frac{1}{x} + \frac{1}{y} \ge \frac{4}{x+y}$. **3**

Question 5 **Use a SEPARATE writing booklet** **Marks**

(a) A mass of 3kg is attached to the vertex of a cone of vertical angle 60^0 by an elastic string of length 1 metre. The mass is moving in a horizontal circle on the curved, frictionless surface of the cone. Acceleration due to gravity is 10 m s^{-2}.

T = Tension in string
N = Normal reaction of the cone surface on the mass.



Not to Scale

(i) If the mass is moving at a speed of 1 m s^{-1}, by resolving forces vertically and horizontally find the values of T and N. **4**

(ii) What is the maximum speed of the particle for it to just remain on the cone's surface and what will be the string's tension at this time? **4**

(b) A mass of 1 kg is allowed to fall under gravity from rest at the surface of a medium in which the retardation on the mass is proportional to the distance fallen (x). In other words, the net force for this motion is $g - kx$ Newtons with the downward direction as positive.

(i) Show that it falls $\frac{2g}{k}$ metres before it becomes stationary. **2**

(ii) Show that the displacement equation in terms of t is given by: **5**

$$x = \frac{g}{k}\left(\sin\left(\frac{2\sqrt{k}\ t - \pi}{2}\right) + 1\right)$$

P.T.O……………………..

Question 6 **Use a SEPARATE writing booklet** **Marks**

(a) Consider the complex number z which satisfies $|z| = 1$.

(i) Using double angle trigonometric identities show that: **2**

$$1 + \cos\alpha + i\sin\alpha \equiv 2\cos\frac{1}{2}\alpha(\cos\frac{1}{2}\alpha + i\sin\frac{1}{2}\alpha).$$

(ii) If $z = \cos\theta + i\sin\theta, -\pi < \theta \le \pi,$ write $1 + z^2$ in terms of $\cos\theta$ and $\sin\theta$. Hence deduce that if in an Argand diagram, points A and B represent z and $1 + z^2$ respectively, then A, B and O are collinear, where O is the origin. State the values of θ such that B lies on the interval OA. **3**

(b) A solid shape has as its base the parabola $y = x^2$ in the XY plane. **5**
Sections taken perpendicular to the axis of the parabola (i.e. perpendicular to the y-axis) are equilateral triangles. Using the method of slicing determine the volume of the solid, if the length of the axis of the parabola is 16cm.

(c) With θ increasing, the point P given by $(r\cos\theta, r\sin\theta)$ is moving in a circular motion about O at a distance of r units from O. Starting with the two equations $x = r\cos\theta$ and $y = r\sin\theta$ prove that the Normal acceleration of P towards O is $r\omega^2$ where ω is the rate of change of θ with respect to time. **5**

Question 7 **Use a SEPARATE writing booklet** **Marks**

(a) Consider the polynomial $P(x) = x^4 - 4x^3 + 5x^2 - 2x - 2$.

(i) Show that the curve $y = P(x)$ has a maximum turning point at (1, -2) and minimum turning points at $x = 1 \pm \frac{1}{2}\sqrt{2}$. Hence deduce from a sketch of the curve that the equation $P(x) = 0$ has two real roots and two non-real roots. **3**

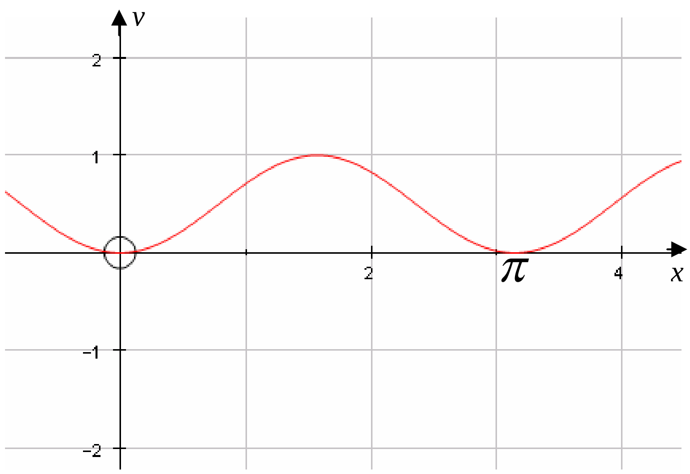
(ii) Explain why the real roots cannot be rational. What do you know about the nature of the non-real roots? **2**

(iii) Given that $1 + i$ is a root of the equation $P(x) = 0$, factor $P(x)$ into two quadratic factors with rational coefficients. Hence find the x-intercepts of the curve $y = P(x)$ and show them on your graph. **3**

(b) (i) Prove that $\int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx.$ **3**

Hence show $\int_0^\pi x\cos 2x\,dx = 0.$

(ii)



The area bounded by the curve $y = \sin^2 x$ and the x-axis between $x = 0$ and $x = \pi$ is rotated through one revolution about the y-axis. By taking the limiting sum of the volumes of cylindrical shells, show that the volume of the solid of revolution is given by $V = 2\pi\int_0^\pi x\,\sin^2 x\,dx$. Hence by using your answer in part (i) or otherwise find the volume of this solid. **4**

P.T.O………………………..

Question 8 **Use a SEPARATE writing booklet** **Marks**

(a) A particle of mass M is projected vertically upward under gravity with speed U in a medium in which the resistance is Mk times the speed, here k is a positive constant. If the particle reaches its greatest height H in time T, show that $U=gT+kH$. (You may assume that the net force is given by $-Mg-Mkv$ with upward direction positive.) **4**

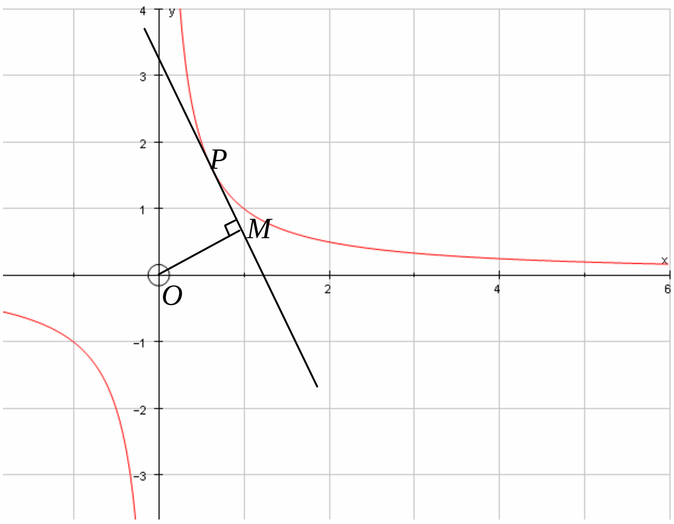
(b) $P(a\cos\theta, b\sin\theta)$ is a point on the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$. PQ is a diameter of the ellipse. The tangent to the ellipse at P meets the vertical through Q at R and the Y-axis at V. O is the origin.

(i) Prove that the tangent at P has equation $\frac{x\cos\theta}{a}+\frac{y\sin\theta}{b}=1$. **2**

(ii) Show that the area of ΔPQR = 4 times the area of ΔPOV. **1**

(iii) Show that the area of ΔPQR is $\frac{2ab}{|\tan\theta|}$ square units. **2**

(c) $P(t, \frac{1}{t})$ is a variable point on the rectangular hyperbola $xy = 1$. M is the foot of the perpendicular from the origin to the tangent to the hyperbola at P.



(i) Show that the tangent to the hyperbola at P has equation $x+t^2y=2t$. **2**

(ii) Find the equation of OM. **1**

(iii) Show that the locus of M as P varies has equation $x^2+y^2=2\sqrt{xy}$. **3**

End of Paper

STANDARD INTEGRALS

$$\int x^n dx = \frac{1}{n+1}x^{n+1}, \quad n \neq -1; \ x \neq 0, \text{ if } n < 0$$

$$\int \frac{1}{x} dx = \ln x, \quad x > 0$$

$$\int e^{ax} dx = \frac{1}{a}e^{ax}, \quad a \neq 0$$

$$\int \cos ax \, dx = \frac{1}{a}\sin ax, \quad a \neq 0$$

$$\int \sin ax \, dx = -\frac{1}{a}\cos ax, \quad a \neq 0$$

$$\int \sec^2 ax \, dx = \frac{1}{a}\tan ax, \quad a \neq 0$$

$$\int \sec ax \tan ax \, dx = \frac{1}{a}\sec ax, \quad a \neq 0$$

$$\int \frac{1}{a^2 + x^2} dx = \frac{1}{a}\tan^{-1}\frac{x}{a}, \quad a \neq 0$$

$$\int \frac{1}{\sqrt{a^2 - x^2}} dx = \sin^{-1}\frac{x}{a}, \quad a > 0, \ -a < x < a$$

$$\int \frac{1}{\sqrt{x^2 - a^2}} dx = \ln\left(x + \sqrt{x^2 - a^2}\right), \quad x > a > 0$$

$$\int \frac{1}{\sqrt{x^2 + a^2}} dx = \ln\left(x + \sqrt{x^2 + a^2}\right)$$

NOTE : $\ln x = \log_e x, \qquad x > 0$