Shore

Student Number:

Set:

Year 12
Mathematics - Extension 2
Trial Examination
2009

General Instructions

* Reading time – 5 minutes
* Working time – 3 hours
* Write using black or blue pen
* Board-approved calculators may be used
* All necessary working should be shown in every question
* A table of standard integrals is attached on the final page

Note: Any time you have remaining should be spent revising your answers.

Total marks - 120

* Attempt Questions 1 - 8
* All questions are of equal value
* Start each question in a new writing booklet
* Write your examination number on the front cover of each booklet to be handed in
* If you do not attempt a question, submit a blank booklet with your examination number and "N/A" on the front cover

DO NOT REMOVE THIS PAPER FROM THE EXAMINATION ROOM

Total marks - 120
Attempt Questions 1 - 8

Answer each question in a SEPARATE writing booklet. Extra writing booklets are available.

Question 1 (15 marks) Use a SEPARATE writing booklet **Marks**

(a) Find the indefinite integrals:

(i) $\int \sec^4 x \, dx$ **2**

(ii) $\int \sqrt{1-x^2} \, dx$ **4**

(b) Consider the definite integral $I_n = \int_0^2 \frac{x^n}{x^3+1} \, dx$.

(i) Show that $I_2 = \frac{2}{3}\log_e 3$. **2**

(ii) Using your knowledge of factorisation and without evaluating more than one integral, show that

$$I_2 - I_1 + I_0 = \log_e 3$$ **2**

(iii) Using a similar approach to that used in (ii), show that

$$I_1 + I_0 = \frac{\pi}{\sqrt{3}}.$$ **3**

(iv) Using the above results or otherwise find the exact value of I_0. **2**

Question 2 **(15 marks)** Use a SEPARATE writing booklet **Marks**

(a) Make neat sketches of the following, showing all intercepts and asymptotes. There is no need to use calculus.

(i) $y = x^2(x-2)(x-3)$ **2**

(ii) $y = \dfrac{1}{x^2(x-2)(x-3)}$ **2**

(iii) $y = \dfrac{x^2}{(x-2)(x-3)}$ **2**

(iv) $y = x\sqrt{(x-2)(x-3)}$ **2**

(v) $y = x^2|x-2|(x-3)$ **2**

(b) Consider the equation $e^{2x} = k\sqrt{x}$.

(i) Explain why this equation has no solutions when $k \le 0$. **1**

(ii) Find the value of k for which the equation has exactly one real solution. **4**

Question 3 **(15 marks)** Use a SEPARATE writing booklet **Marks**

(a) Given $z = 1 - 2i$ is a complex root of the quadratic equation $z^2 + (1+i)z + k = 0$, find the other root and the value of k. **3**

(b) Find all complex numbers $z = a + bi$, where a and b are real such that $|z|^2 - iz = 16 - 2i$. **3**

(c) Consider all complex numbers z such that $\arg\left(\frac{z-1}{z-i}\right) = \frac{\pi}{4}$

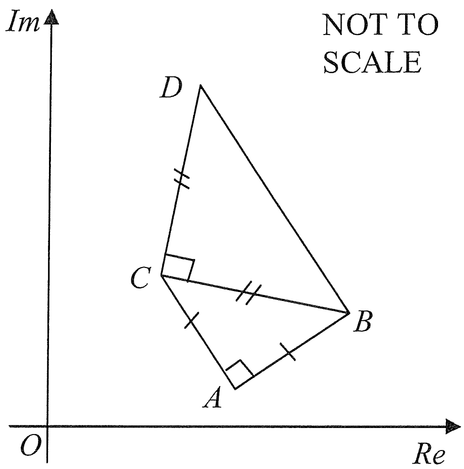
(i) Make a neat sketch of the locus of z showing important features. **2**

(ii) Determine the exact maximum value of $|z|$. **1**

(iii) Determine (in radians correct to 3 significant figures) the maximum value of $\arg(z+1)$. **3**

Question 3 continues on page 5

(d)



In the diagram, the points A, B, C and D represent the complex numbers z_1, z_2, z_3 and z_4 respectively. Both ΔABC and ΔBCD are right angled isosceles triangles as shown.

(i) Show that the complex number z_3 can be written as

$$z_3 = (1-i)z_1 + iz_2 .$$ **1**

(ii) Hence express the complex number z_4 in terms of z_1 and z_2, giving your answer in simplest form. **2**

End of Question 3

Question 4 **(15 marks)** Use a SEPARATE writing booklet **Marks**

(a) Use Mathematical Induction to prove De Moivre's Theorem, ie

$(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$ for all positive integers n. **4**

(b) The equation $x^3 + 3px - 1 = 0$, where p is real, has roots α, β and γ.

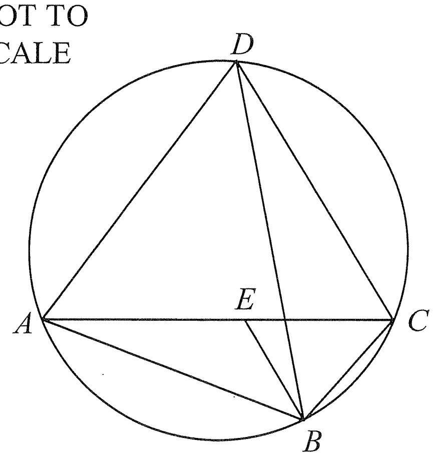
(i) Show that the monic cubic equation, with coefficients in terms of p, whose roots are α^2, β^2 and γ^2 is

$$y^3 + 6py^2 + 9p^2y - 1 = 0.$$ **2**

(ii) Hence or otherwise obtain the monic cubic equation, with coefficients in terms of p, whose roots are $\frac{\beta\gamma}{\alpha}$, $\frac{\gamma\alpha}{\beta}$ and $\frac{\alpha\beta}{\gamma}$. **3**

(c)

NOT TO SCALE



The figure shows a cyclic quadrilateral $ABCD$ with diagonals AC and BD.

E is a point on AC such that $\angle ABE = \angle DBC$.

Make a neat copy of the diagram in your answer booklet.

(i) Prove that $\Delta ABE \,|||\, \Delta DBC$. **2**

(ii) Prove that $\Delta ABD \,|||\, \Delta EBC$. **2**

(iii) Hence prove Ptolemy's Theorem, which is:

$$AB.DC + AD.BC = AC.BD$$ **2**

Question 5 **(15 marks)** Use a SEPARATE writing booklet **Marks**

(a) Determine the eccentricity of the ellipse with equation $\frac{x^2}{16}+\frac{y^2}{25}=1$ and then make a neat sketch of the curve, clearly showing the coordinates of the foci and the equations of the directrices. **4**

(b) A point $P(a\sec\theta, b\tan\theta)$ lies on the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$.

The line through P perpendicular to the x-axis meets an asymptote at Q and the normal at P meets the x-axis at N.

(i) Make a neat sketch illustrating the information above. **1**

(ii) Show that the equation of the normal at P is

$$\frac{ax}{\sec\theta}+\frac{by}{\tan\theta}=a^2+b^2.$$ **2**

(iii) Show that QN is perpendicular to the asymptote. **2**

(c) $P\left(p,\frac{1}{p}\right)$ and $Q\left(q,\frac{1}{q}\right)$ are two variable points on the rectangular hyperbola $xy=1$ such that the chord PQ passes through the point $A(0,2)$. M is the midpoint of PQ.

(i) Show that PQ has equation $x+pqy-(p+q)=0$ and hence deduce that $p+q=2pq$. **3**

(ii) You may assume that the tangent to $xy=1$ at the point $(1,1)$ passes through A. Determine the locus of M, being sure to state any restrictions on the domain. **3**

Question 6 **(15 marks)** Use a SEPARATE writing booklet **Marks**

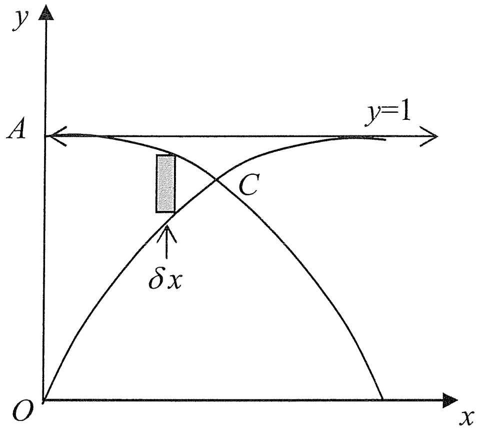
(a) The base of a solid is the region enclosed by the circle $x^2 + y^2 = 4$. Any cross sections of the solid formed by a plane perpendicular to the x-axis are equilateral triangles. Find the exact volume of the solid. **4**

(b) (i) Make a neat sketch of the region enclosed between the curve $y = (x-3)^2$ and the line $3x + y - 9 = 0$. Be sure to mark in the points of intersection. **2**

(ii) The shaded region in (i) is rotated about the line $x = 3$. Use the method of cylindrical shells to find the exact volume of the solid generated. **3**

Question 6 continues on page 9

(c) The diagram below shows part of the graphs of $y = \cos x$ and $y = \sin x$. The graph of $y = \cos x$ meets the y-axis at A, and C is the first point of intersection of the two graphs to the right of the y-axis. The region OAC is to be rotated about the line $y = 1$.



NOT TO SCALE

(i) Write down the coordinates of the point C. **1**

(ii) The shaded strip of width δx shown in the diagram is rotated about the line $y = 1$. Show that the volume δV of the resulting slice is given by

$$\delta V = \pi\left(2\cos x - 2\sin x - \cos 2x\right)\delta x\,.$$ **2**

(iii) Hence find the exact volume of the solid formed when the region OAC is rotated about the line $y = 1$. **3**

End of Question 6

Question 7 **(15 marks)** Use a SEPARATE writing booklet **Marks**

(a) (i) Prove that $\tan^{-1}(n+1)-\tan^{-1}(n)=\cot^{-1}(1+n+n^2)$ **2**

(ii) Hence find the sum of the finite series

$$\cot^{-1}(3)+\cot^{-1}(7)+\cot^{-1}(13)+.....+\cot^{-1}(1+n+n^2)$$

Give your answer in simplest form. **2**

(b) You are given that for the complex number $z=\cos\theta+i\sin\theta$ and for positive integers n, the following results are true:

$$z^n+\frac{1}{z^n}=2\cos n\theta \quad \text{and} \quad z^n-\frac{1}{z^n}=2i\sin n\theta$$

(i) Expand $\left(z+\frac{1}{z}\right)^4+\left(z-\frac{1}{z}\right)^4$ and hence show that

$$4\cos^4\theta+4\sin^4\theta=\cos 4\theta+3$$ **3**

(ii) By letting $x=\cos\theta$, show that the equation

$8x^4+8\left(1-x^2\right)^2=7$ has roots $x=\pm\cos\frac{\pi}{12},\ \pm\cos\frac{5\pi}{12}$. **3**

(iii) Deduce that $\cos\frac{\pi}{12}\cos\frac{5\pi}{12}=\frac{1}{4}$ and $\cos\frac{\pi}{12}+\cos\frac{5\pi}{12}=\sqrt{\frac{3}{2}}$. **3**

(iv) Hence or otherwise express $\cos\frac{\pi}{12}$ in surd form. **2**

Question 8 **(15 marks)** Use a SEPARATE writing booklet **Marks**

(a) Six letters are chosen from the word AUSTRALIA. These six letters are then placed alongside one another to form a six letter arrangement. Find the number of distinct six letter arrangements which are possible, considering all choices. **4**

(b) It is given that for three positive real numbers a, b and c,

$$\frac{a+b+c}{3} \geq \sqrt[3]{abc}$$

If we also know that $a+b+c=1$, prove that

(i) $\dfrac{1}{abc} \geq 27$ **1**

(ii) $\dfrac{1}{a}+\dfrac{1}{b}+\dfrac{1}{c} \geq 9$ **2**

(iii) $(1-a)(1-b)(1-c) \geq 8abc$ **2**

(c) Let $I_n = \int_0^{\frac{\pi}{2}} \sin^n x \, dx$.

(i) Show that $I_n = \left(\dfrac{n-1}{n}\right) I_{n-2}$ for $n \geq 2$. **3**

(ii) Hence show that $\int_0^{\frac{\pi}{2}} \sin^{2n} x \, dx = \dfrac{\pi (2n)!}{2^{2n+1} (n!)^2}$ **3**

END OF EXAM

STANDARD INTEGRALS

$$\int x^n dx = \frac{1}{n+1}x^{n+1}, \quad n \neq -1; \quad x \neq 0 \text{ if } n < 0$$

$$\int \frac{1}{x}dx = \ln x, \quad x > 0$$

$$\int e^{ax} dx = \frac{1}{a}e^{ax}, \quad a \neq 0$$

$$\int \cos ax dx = \frac{1}{a}\sin ax, a \neq 0$$

$$\int \sin ax dx = -\frac{1}{a}\cos ax, a \neq 0$$

$$\int \sec^2 ax dx = \frac{1}{a}\tan ax, a \neq 0$$

$$\int \sec ax \tan ax dx = \frac{1}{a}\sec ax, a \neq 0$$

$$\int \frac{1}{a^2 + x^2}dx = \frac{1}{a}\tan^{-1}\frac{x}{a}, a \neq 0$$

$$\int \frac{1}{\sqrt{a^2 - x^2}}dx = \sin^{-1}\frac{x}{a}, a > 0, -a < x < a$$

$$\int \frac{1}{\sqrt{x^2 - a^2}}dx = \ln(x + \sqrt{x^2 - a^2}), x > a > 0$$

$$\int \frac{1}{\sqrt{x^2 + a^2}}dx = \ln(x + \sqrt{x^2 + a^2})$$

Note: $\ln x = \log_e x, x > 0$