

St. George Girls' High School

1999

TRIAL HIGHER CERTIFICATE EXAMINATION



MATHEMATICS

4 Unit

TIME ALLOWED: 3 HOURS
(Plus 5 minutes' reading time)

INSTRUCTIONS TO CANDIDATES:

1. All questions may be attempted
2. All necessary working must be shown
3. Marks may be deducted for careless or poorly presented work
4. Begin each question on a NEW PAGE
5. A list of standard integrals is included at the end of this paper
6. The mark allocated for each question is listed at the side of the question

Students are advised that this is a Trial Examination only and does not necessarily reflect the content or format of the Higher School Certificate Examination.

QUESTION 1 (15 marks)

a) Evaluate $\int_e^{e^2} \frac{dx}{x(\ln x)^3}$ 3

b) i) Show that $\frac{x^3}{1+x^2} = x - \frac{x}{1+x^2}$ 4

ii) Find $\int x^2 \tan^{-1} x \, dx$

c) Evaluate $\int_0^{\pi/2} \frac{dx}{1 + \sin x + \cos x}$ 3

d) $I_n = \int_0^{\pi/2} \cos^n x \, dx \quad n \geq 0$ 5

i) Show that $I_n = \frac{n-1}{n} I_{n-2}$

ii) Hence evaluate $\int_0^{\pi/2} \cos^4 x \, dx$

QUESTION 2 (15 marks)

a) The points P and Q represent the complex numbers $3 + 4i$ and $1 - 2i$ respectively. 3

i) Plot the points P and Q on an Argand diagram and plot the point R such that $POQR$ is a parallelogram.

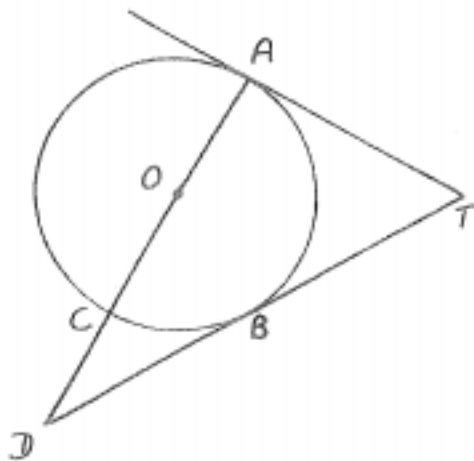
ii) What complex number does R represent?

b) On separate diagrams sketch the locus specified by: 4

i) $\arg(z - 2) = \arg(z + i)$

ii) $|z - 2 - 3i| = 2$

c) 8



From an external point T , two tangents TA and TB are drawn to touch a circle with centre O at A and B respectively.

$\hat{A}TB$ is acute. The diameter AC produced meets TB produced at D .

i) Prove that $\hat{C}BD = \frac{1}{2} \hat{A}TB$

ii) Prove that $\triangle ABC$ is similar to $\triangle TBO$

iii) Deduce that $BC \cdot OT = 2 \cdot (OA)^2$

QUESTION 3 (15 marks)

The hyperbola H has Cartesian equation $\frac{x^2}{16} - \frac{y^2}{9} = 1$

- a) Find: i) its eccentricity
ii) the coordinates of its foci, S_1 and S_2
iii) the equations of its directrices
iv) the equations of its asymptotes

4

- b) Sketch H , clearly showing any intercepts with the coordinate axes and the details found in (a).

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- c) P is the point $(4 \sec \theta, 3 \tan \theta)$

3

- i) Show that P lies on H .
ii) Show that the tangent to H at P has equation

$$\frac{x \sec \theta}{4} - \frac{y \tan \theta}{3} = 1$$

- d) The tangent at P cuts the asymptotes at the points C and D .

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- i) Prove that $CP = DP$
ii) Show that $OC \cdot OD = 25$
iii) Hence show that the area of $\triangle OCD$ is independent of the position of P on H .

QUESTION 4 (15 marks)

- a) i) On the same set of axes sketch the graphs of

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$$y = |x + 1| \text{ and } y = |x^2 - 1|$$

- ii) Hence, or otherwise, sketch the graph of:

$$f(x) = |x + 1| + |x^2 - 1|$$

- iii) For which values of x is $f(x) > 2$

- iv) Discuss the differentiability of $f(x)$ at $x = 1$

- b) i) Show that the area of an isosceles right angled triangle with

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hypotenuse h is $\frac{h^2}{4}$

- ii) The base of a solid is the region enclosed by the curve $y = 9 - x^2$ and the x -axis. Each cross-section perpendicular to the y -axis is an isosceles right angled triangle with hypotenuse lying in the base. Find the volume of the solid.

QUESTION 5 (15 marks)

a) $P(x) = x^3 + bx + 1$, b is a real number

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i) If the roots of $P(x) = 0$ are α, β, γ find the polynomial equation with roots of:

(α) $3\alpha, 3\beta, 3\gamma$

(β) $\alpha^2, \beta^2, \gamma^2$

(γ) $\alpha + \beta - \gamma, \beta + \gamma - \alpha, \gamma + \alpha - \beta$

ii) For which values of b will $P(x) = 0$ have two complex roots.

b) $z = \frac{\sqrt{3}}{2} + \frac{1}{2}i$

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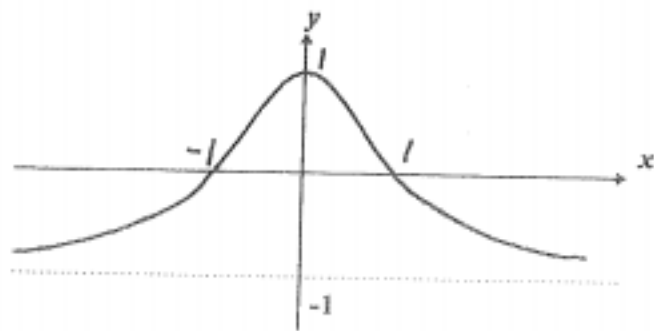
i) Express z in modulus-argument form

ii) Find the smallest positive integer n such that:

$$1 + z + z^2 + \dots + z^{n-1} = 0$$

QUESTION 6

- a) The graph of $f(x) = \frac{1-x^2}{1+x^2}$ is shown below.



On separate diagrams, sketch the graphs of the following functions, clearly showing all intercepts and asymptotes.

- i) $y = \frac{1}{f(x)}$ 2
- ii) $y = [f(x)]^2$ 2
- iii) $y^2 = f(x)$ 2

- b) The acceleration $\ddot{x} \text{ cm/s}^2$ of a particle moving along the x -axis is given by: 5

$$\ddot{x} = f(x) \quad (\text{where } x \text{ is measured in cm})$$

where $f(x)$ is as shown above.

- i) If the particle was initially at rest at the origin, find its velocity at $x = 1$.
- ii) This particle next stops at $x = k$.
Show that $2 < k < 3$.
- iii) Using an appropriate first approximation and one application of Newton's method, find a value for k .

QUESTION 7

- a) An object of mass 100 kg experiences air resistance of $\frac{v^2}{20}$ Newtons, 8
 where $v \text{ ms}^{-1}$ is the velocity of the object. This object falls from rest from a height of h metres above the ground. Let x metres be the distance of the object from its starting point and the acceleration due to gravity be 9.8 ms^{-2} .

- i) Show that $\ddot{x} = 9.8 - \frac{v^2}{2000}$
- ii) Find the terminal velocity of the object
- iii) If the object reaches a velocity of 60% of its terminal velocity at the instant it hits the ground calculate the value of h correct to one decimal place.

- b) i) Show that $\int_0^2 xe^x dx = e^2 + 1$ 7

- ii) Draw a neat sketch of the area bounded by the y axis, the line $x = 2$ and the curves $y = e^x$ and $y = x^2$
- iii) The area in (ii) is rotated about the line $x = 3$. Using cylindrical shells or otherwise show that the volume of the solid generated is given by:

$$V = 2\pi \int_0^2 (3-x)(e^x - x^2) dx$$

- iv) Hence find the exact value of V .

QUESTION 8 (15 marks)

a) i) Express $\sin z + \cos z$ in the form $R \sin(z + \alpha)$ where:
 $R > 0$ and $0 < \alpha < \frac{\pi}{2}$ 8

ii) Given that $y = e^x \sin x$ show that:

$$\frac{dy}{dx} = \sqrt{2} e^x \sin\left(x + \frac{\pi}{4}\right)$$

iii) Prove by mathematical induction that for $y = e^x \sin x$ and n a positive integer

$$\frac{d^n y}{dx^n} = 2^{\frac{n}{2}} e^x \sin\left(x + \frac{n\pi}{4}\right)$$

where $\frac{d^n y}{dx^n}$ denotes the n th derivative of y with respect to x

b) Given that $P(x) = x^4 + 2x^3 - 2x^2 + 8$ has a zero of multiplicity 2, find
all solutions of $P(x) = 0$ over the field of complex numbers 4

c) Prove that: $\binom{n+2}{r} = \binom{n}{r-2} + 2\binom{n}{r-1} + \binom{n}{r}$ 3

The following list of standard integrals may be used:

$$\int x^n dx = \frac{1}{n+1} x^{n+1}, \quad n \neq -1; \quad x \neq 0, \quad \text{if } n < 0$$

$$\int \frac{1}{x} dx = \ln x, \quad x > 0$$

$$\int e^{ax} dx = \frac{1}{a} e^{ax}, \quad a \neq 0$$

$$\int \cos ax dx = \frac{1}{a} \sin ax, \quad a \neq 0$$

$$\int \sin ax dx = -\frac{1}{a} \cos ax, \quad a \neq 0$$

$$\int \sec^2 ax dx = \frac{1}{a} \tan ax, \quad a \neq 0$$

$$\int \sec ax \tan ax dx = \frac{1}{a} \sec ax, \quad a \neq 0$$

$$\int \frac{1}{a^2 + x^2} dx = \frac{1}{a} \tan^{-1} \frac{x}{a}, \quad a \neq 0$$

$$\int \frac{1}{\sqrt{a^2 - x^2}} dx = \sin^{-1} \frac{x}{a}, \quad a > 0, \quad -a < x < a$$

$$\int \frac{1}{\sqrt{x^2 - a^2}} dx = \ln(x + \sqrt{x^2 - a^2}), \quad x > a > 0$$

$$\int \frac{1}{\sqrt{x^2 + a^2}} dx = \ln(x + \sqrt{x^2 + a^2}).$$

Note: $\ln x = \log_e x, \quad x > 0$