SYDNEY TECHNICAL HIGH SCHOOL

TRIAL HIGHER SCHOOL CERTIFICATE EXAMINATION - 1991

3 UNIT MATHEMATICS

NAME: ______________________ CLASS: ______________

Time allowed: 2 hours

Instructions: All questions may be attempted.
All questions are of equal value.
All necessary working must be shown.
Full marks may not be awarded for careless or badly arranged work.
Standard integrals are printed on the last page.
Start each question on a new page.
This exam is to form part of your H.S.C. assessment.

Question	1	2	3	4	5	6	7
Mark							

QUESTION 1

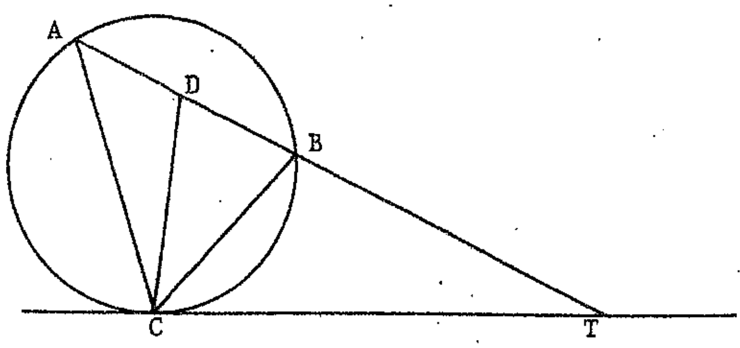
a) Sketch, without the use of calculus, $y = x(x - 2)(x - 3)^2$

b) Given that $\log_8 2 = \log_x 5$, find x.

c) Differentiate $e^x \operatorname{Cos} x$

d) Solve for x : $\left|\frac{2}{x-3}\right| > 5$

e) ABC is a triangle inscribed in a circle, the tangent at C meets AB at T. The bisector of angle ACB cuts AB at D.



i) Draw a neat diagram showing the above information.

ii) Prove TC = TD

QUESTION 2 (Start a new page)

a) Evaluate

i) $\int_0^{\frac{\pi}{4}} \text{Cos } 2x\,dx$

ii) $\int_0^{\frac{1}{6}} \frac{dx}{\sqrt{1 - 9x^2}}$

b) Use the substitution $u = 25 - x^2$ to evaluate $\int_3^4 \frac{x\,dx}{\sqrt{25 - x^2}}$

c) The sides of a cube are increasing at the rate of 2 mm/sec. Find the rate of increase of the volume of the cube when the sides are each 10 cm long.

QUESTION 3 (Start a new page)

a) Sketch the graph of $y = \tan^{-1} x$

State the domain and range of the function.

b) Differentiate $\tan^{-1}\sqrt{x^2 - 1}$

c) i) Show that $\text{Sin } 3x = 3\,\text{Sin } x - 4\,\text{Sin}^3 x$

ii) Hence or otherwise evaluate $\int_0^{\frac{\pi}{3}} \text{Sin}^3 x\,dx$

QUESTION 4 (Start a new page)

a) O is the centre of a circle, CB is a chord parallel to a radius OA. OB cuts AC at a point K inside the circle.

i) Draw a neat sketch on your answer sheet showing the given information.

ii) Prove that angle AKB = 3 x angle ACB

Question 4 continued over page

b) A sprinkler projects water from ground level with a velocity of V m/s at an angle of elevation of $\Theta°$.

i) Derive expressions for the horizontal and vertical components of displacement from the point of projection after t seconds starting with the equations of motion $\ddot{x} = 0$, $\ddot{y} = -10$

ii) Show that the horizontal range of the projected water is given by $\frac{V^2 \sin 2\Theta}{10}$.

iii) If the sprinkler has been designed to project water at all angles of elevation from 30° to 75° at the same time, calculate the range of distances from the sprinkler in which water will fall if the velocity of projection is 20 m/s.

QUESTION 5 (Start a new page)

a) Prove the following by mathematical induction, where n is a positive integer

$$1 \times 3 + 2 \times 3^2 + \ldots\ldots + n \times 3^n = \frac{(2n - 1).3^{n+1} + 3}{4}$$

b) i) Find the equation of the tangent to the parabola $x^2 = 4ay$ at the point $P(2ap, ap^2)$

ii) If the tangent cuts the x axis at R and the y axis at T, find the coordinates of R and T.

iii) If ORQT is a rectangle, where O is the vertex of the parabola, find the locus of Q as P moves around the parabola.

ESTION 6 (Start a new page)

a) Given $x^3 - 8x^2 + 9x + k = 0$ determine the integral value of k if one of the roots is twice the other.

b) A monic polynomial of degree 3 is known to have only one real root, $x = 3$. Write a general expression for the polynomial equation stating any conditions which apply to the constants you use.

c) A particle moves in a straight line so that its distance from the original at time t is x.

i) Show that $\frac{d^2x}{dt^2} = \frac{d}{dx}(\frac{1}{2}v^2)$ where v denotes velocity.

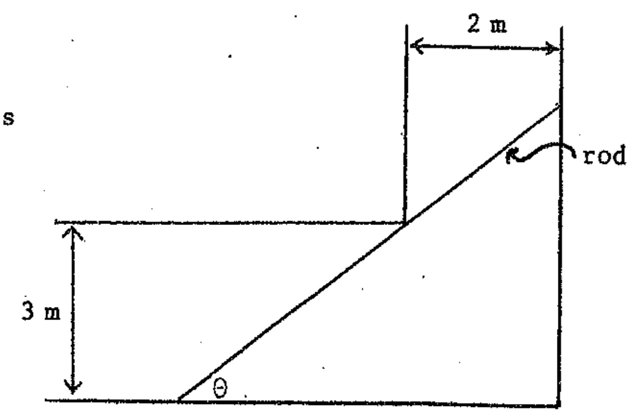
ii) If $\frac{dx}{dt} = \sin x$, find an expression for acceleration as a function of displacement.

QUESTION 7 (Start a new page)

a) Using the t results or otherwise show that $\cot\theta + \tan\frac{\theta}{2} = \operatorname{cosec}\theta$

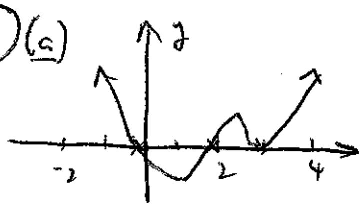
b) Two corridors 3 metres wide and 2 metres wide respectively meet at right angles.

Find the length of the longest thin straight rod that will pass horizontally around the corner. (Assume width of rod negligible)



STHS TRIAL 1991
3 UNIT
SOLUTIONS

1 (a)



(b) Let $y = \log_8 2$

$\therefore 2 = 8^y$

$\therefore y = 1/3$

$\therefore \log_x 5 = 1/3$

$x^{1/3} = 5$

$\underline{x = 125}$

(c) $\cos x\, e^x + e^x(-\sin x)$

$= e^x(\cos x - \sin x)$

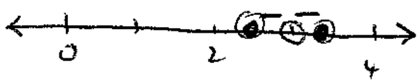
(d) $\underline{\text{c.v. } x \neq 3}$

$\frac{2}{x-3} = 5$ or $\frac{2}{x-3} = -5$

$5x - 15 = 2$ $-5x + 15 = 2$

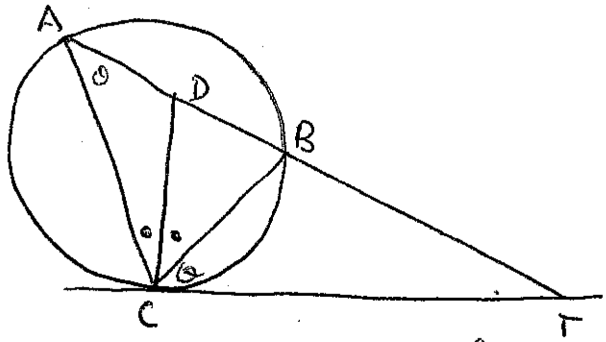
$5x = 17$ $5x = 13$

$x = 17/5$ $x = 13/5$



$13/5 \leq x \leq 17/5 \quad x \neq 3$

(e)



Let $\angle TCB = \theta$ and $\angle ACD = \angle BCD = x^\circ$

$\therefore \angle CAB = \theta$ (angle in the alt. segment)

$\therefore \angle BTC = 180 - (2\theta + 2x)$ (angle sum of $\triangle ACT$)

$\therefore \angle CDT = 180 - [180 - 2\theta - 2x + x + \theta]$

$= \theta + x$

$= \angle DCT$

$\therefore \triangle DCT$ is isosceles

$\therefore DT = CT$

2 (a) $\int_0^{\pi/4} \cos 2x\, dx = \left.\frac{1}{2}\sin 2x\right]_0^{\pi/4}$

$= 1/2 \sin \pi/2 - 1/2 \sin 0$

$= 1/2$

(ii) $\int_0^{1/6} \frac{dx}{\sqrt{1-9x^2}} = \frac{1}{3}\int_0^{1/6} \frac{3}{\sqrt{1-9x^2}}\, dx$

$= \left.1/3 \sin^{-1} 3x\right]_0^{1/6}$

$= 1/3(\sin^{-1} 1/2 - \sin^{-1} 0)$

$= 1/3 \cdot \pi/6$

$= \pi/18$

(b) $u = 25 - x^2$

$\frac{du}{dx} = -2x$

$dx = \frac{du}{-2x}$

$\int \frac{x\, dx}{\sqrt{25-x^2}} = \int \frac{x \cdot \frac{du}{-2x}}{\sqrt{u}}$

$= -1/2 \int \frac{du}{\sqrt{u}}$

$= -u^{1/2}$

$\therefore \int_3^4 \frac{x\, dx}{\sqrt{25-x^2}} = \left[-\sqrt{25-x^2}\right]_3^4$

$= -\sqrt{9} + \sqrt{16}$

$= -3 + 4$

$= 1$

(c) $\frac{dx}{dt} = 2 \qquad V = x^3$

$\frac{dV}{dx} = 3x^2$

$\frac{dV}{dt} = \frac{dV}{dx} \cdot \frac{dx}{dt}$

$= 3x^2 \cdot 2$

$\underline{\text{At } x = 100}$

$\frac{dV}{dt} = 3(100)^2 \cdot 2$

$= 60{,}000 \text{ mm}^3/\text{sec}$

(5) For $n = 1$ LHS $= 3$ RHS $= \dfrac{(2-1)(3^2)+3}{4}$

$= 3.$

<u>assume true for $n = k$.</u>

$1 \times 3 + 2 \times 3^2 + \ldots + k \times 3^k = \dfrac{(2k-1)3^{k+1}+3}{4}$

TARGET: $\dfrac{(2k+1)3^{k+2}+3}{4}$

<u>For $n = k+1$</u>

LHS $= 1 \times 3 + 2 \times 3^2 + \ldots + (k)3^k + (k+1)3^{k+1}$

$= \dfrac{(2k-1)3^{k+1}+3}{4} + (k+1)3^{k+1}$

$= \frac{1}{4}3^{k+1}\left[(2k-1) + 4(k+1)\right] + 3/4$

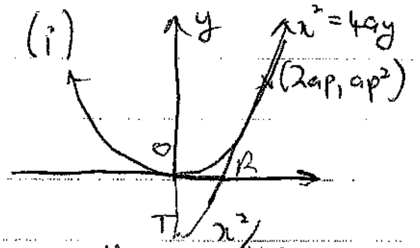
$= \frac{1}{4}3^{k+1}\left[6k+3\right] + 3/4$

$= \frac{1}{4}3^{k+1}\,3\left[2k+1\right] + 3/4$

$= \dfrac{3^{k+2}(2k+1)+3}{4}$

which is of the same form.

(<u>b</u>) (i)



$y = x^2/4a$

$dy/dx = 2x/4a$

$= x/2a$

<u>At $(2ap, ap^2)$</u> $m = p$

$y - ap^2 = p(x - 2ap)$

$y - ap^2 = px - 2ap^2$

$y = px - ap^2$

(<u>ii</u>) R is $(ap, 0)$

T is $(0, -ap^2)$

(<u>iii</u>) Q is $(ap, -ap^2)$

$-px = y$

ie. st line $y + px = 0$

6 (a) $x^3 - 8x^2 + 9x + k = 0$

$\alpha, 2\alpha, \beta$

Sum of roots $= 8/1 = 3\alpha + \beta$

$\beta = 8 - 3\alpha$

Sum of roots (x2)

$2\alpha^2 + 3\alpha\beta = 9$ (1)

Product of roots

$2\alpha^2\beta = -k$ (2)

In (1) $2\alpha^2 + 3\alpha(8 - 3\alpha) = 9$

$2\alpha^2 + 24\alpha - 9\alpha^2 = 9$

$7\alpha^2 - 24\alpha + 9 = 0$

$(7\alpha - 3)(\alpha - 3) = 0$

$\therefore \alpha = 3$ and $\beta = -1$

$\therefore$ Product of roots is

$2(3)^2(-1) = -k$

$k = 18$

(b) $ax^3 + bx^2 + cx + d = P(x)$

$27 + 9b + 3c + d = 0$

(c) (i) Bookwork

(ii) $\frac{dx}{dt} = \sin x$

$\frac{1}{2}v^2 = \frac{1}{2}\sin^2 x$

$\therefore a = \frac{d}{dx}\left(\frac{1}{2}\sin^2 x\right)$

$= \sin x \cos x$

7

$\cot\theta + \tan\theta/2$

$= \frac{1 - \tan^2\theta/2}{2\tan\theta/2} + \tan\theta/2$

$= \frac{1 - t^2}{2t} + t$

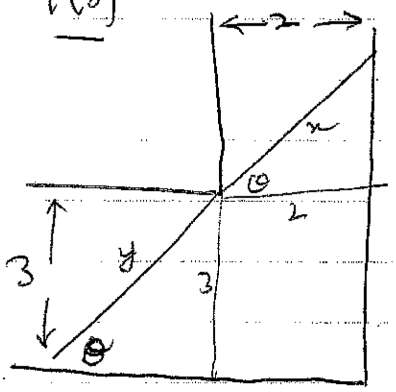
$= \frac{1 - t^2 + 2t^2}{2t}$

$= \frac{1 + t^2}{2t}$

$= 1/\sin\theta$

$= \operatorname{cosec}\theta$

7(b)



$y/3 = \operatorname{cosec}\theta$ $\quad x/2 = \sec\theta$

$l = x + y$

$= 3\operatorname{cosec}\theta + 2\sec\theta$

$\frac{dl}{d\theta} = -3(\sin\theta)^{-2}\cdot\cos\theta - 2(\cos\theta)^{-2}(-\sin\theta)$

$= \frac{-3\cos\theta}{\sin^2\theta} + \frac{2\sin\theta}{\cos^2\theta}$

$=$